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Locally U(1)×U(1) Symmetric Cosmological Models: Topology and Dynamics

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ABSTRACT

We show examples which reveal influences of spatial topologies to dynamics, using a class of spatially *closed* inhomogeneous cosmological models. The models, called the *locally U(1)×U(1) symmetric models* (or the *generalized Gowdy models*), are characterized by the existence of two commuting spatial *local* Killing vectors. For systematic investigations we first present a classification of possible spatial topologies in this class. We stress the significance of the locally homogeneous limits (i.e., the Bianchi types or the 'geometric structures') of the models. In particular, we show a method of reduction to the natural reduced manifold, and analyze the equivalences at the reduced level of the models as dynamical models. Based on these fundamentals, we examine the influence of spatial topologies on dynamics by obtaining translation and reflection operators which commute with the dynamical flow in the phase space.

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1 Introduction

Inhomogeneous spacetime models generally inherit from the full relativistic dynamics the strong nonlinearity in straightest ways, so that a moderately simple inhomogeneous model can be a good testing ground to obtain an insight toward understandings of generic properties of the relativistic dynamics. One of the best-known such models is the *Gowdy model*, where, first of all, two commuting spatial Killing vectors are assumed. By this assumption the spatial manifold is simplified to a one dimensional reduced manifold. Second, it is assumed that the spatial manifold is compact without boundary, i.e., *closed*. This assumption is favorable in view of the fact that no ambiguities occur in the boundary conditions once a topology is specified. A closed space is physically natural, as well, due to, e.g., the finiteness of the gravitational action.

A striking property resulting from these two assumptions is the fact that the possible spatial topologies are restricted. When Gowdy models first appeared in 1971 [1] as global generalizations of cylindrical plane wave solutions, only two spatial topologies, $S^2 \times S^1$ and S^3, were considered. After a few years from the first paper, Gowdy [2] became aware of the fact that any closed three dimensional Riemannian manifold which admits two commuting Killing vectors is homeomorphic to one of T^3, $S^2 \times S^1$, S^3 or a lens space $L(p, q)$. So, he added T^3 model to his consideration. (Since lens spaces are finitely covered by the sphere S^3, he argued little about lens space models.) This class of solutions or cosmological models consists of the original version of the Gowdy models.

Recently, it has been pointed out [3, 4] that a natural generalization is possible, where the two commuting Killing vectors are *local*, i.e., the Killing vectors are defined in a neighborhood of every point but are not necessarily globally defined. (More precise definition of local Killing vectors will be presented in the next section.) This generalization gives us a set of new favorable models to test the dynamical properties of relativity, especially, as we will see, in connection with the spatial topology.

Part of the motivation of this work comes from a plan to systematically investigate how the spatial topology influence the dynamics of a spacetime. In fact, the relativistic dynamics of a spacetime often seems to be influenced by its spatial topology. For example, we know the recollapsing conjecture [5–7], where it is claimed that the well-known recollapsing property of a positive curvature (topologically S^3) homogeneous and isotropic cosmological model continues to hold also when the spacetime is inhomogeneous (if only appropriate energy and pressure conditions are fulfilled). Thus we may interpret the recollapsing property is a direct consequence derived from the spatial topology.

As another example, note (e.g., [8]) that the dynamics of a vacuum homogeneous universe can be well characterized by the "potential" (given by the spatial scalar curvature), which can be determined if we know the Bianchi type. The local dynamics of a compactified (locally) homogeneous model can also be understood in the same manner (see [9–11]). Moreover in most cases, the corresponding Bianchi type is unique if the spatial manifold is closed. Thus we can say that the spatial topology determines the dynamical behavior of the universe at least at the locally homogeneous limit.

How about when the spacetime becomes inhomogeneous? Can one, as in the locally homogeneous cases, characterize the dynamics of an inhomogeneous universe by the spatial topology? When the spacetimes become inhomogeneous distinctions in local properties will obviously dis-

appear, as the Einstein field equations can be written in the same form, irrespective of the spatial topology. However, the boundary conditions may be different in general. So, we can naturally expect that the distinctions in the dynamical properties due to the differences in the spatial topologies manifest themselves in their *global, discrete, statistical, or average* properties. (Note that the recollapsing property mentioned above itself is also regarded as a global property.) It seems fascinating to find such properties. Also, such an investigation might provide us with some clues to the topology of our universe.

In this paper we show, after some preparations availing ourselves of the generalized Gowdy models, an example of such properties using those models. More precise contents and contributions should be the following.

We first reveal all the possible spatial topologies of the generalized Gowdy models and classify them. In particular, we categorize all the models into two kinds, then subdivide each of them into finite types each of which has a correspondence to a particular type of locally homogeneous manifolds. Although we show the best classification we have, which will require further considerations (in fact, the number of the possible topologies is infinite), the set of the (finite) types mentioned above will be found to present the most relevant set of classes in an appropriate sense. (To be precise, however, some of the classes can have a discrete parameter.) While by our systematic analysis we find no new topologies other than those in the literature [1–4], one of our contributions is that we first asserts that those models do exhaust all the possible ones.

Then we will concentrate on the models belonging to one kind, called the first kind, where each spatial manifold is a T^2-bundle over S^1. For these models we argue how we can represent the metric, and discuss the question of *natural reduction* (to S^1 as the spatial part). As a result of the reduction, we find that some models give rise to the same reduced Einstein equations with the same boundary conditions for their metric functions. We say that such two models are *dynamically equivalent* to each other. The dynamical equivalence greatly diminishes the number of representative models (or topologies) we should consider. The basic idea for the reductions is the same as the one presented in Ref. [3], while the dynamical equivalence is first introduced in this paper. All of these settings are an indispensable step for the subsequent study of the models.

Finally, motivated by the reason explained above, we consider varieties of spatial *translation* and *reflection* symmetries, and then ask if these symmetries imposed on initial data sets are preserved in time or not for each reduced model. In fact, we find remarkable differences in the reflections.

These are done in Secs.2 to 4. Section 5 is devoted to a summary and comments. In particular, comments on AVTD behavior and the local U(1) models are made. Appendix A gives a summary of the standard description of the Gowdy models with generalizations. Appendix B gives a summary of a calculation for the compactification of Sol geometry.

We adopt the abstract index notation [12], that is, we use small Latin indices $a, b, \cdots$ *not* to denote components but to represent the type of a tensor explicitly. To denote components we use Greek indices $\mu, \nu \cdots$ or capital Latin indices $A, B, \cdots$. Conventionally we write tilde to denote a metric on a universal cover like $\tilde{\mathrm{g}}_{ab}$, while a metric on a quotient space is simply written as g_{ab}. In the component representation, however, these metrics are possibly represented in the same way like $\mathrm{g}_{\mu\nu}$. We assume that all the spatial three-manifolds are orientable in this paper.

2 Topologies and Geometries of Locally $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetric spaces and spacetimes

Let (M, h_{ab}) be a Riemannian manifold . Suppose for any point p in M there exit neighborhood U which admits Killing vector fields, but these Killing vectors are not necessarily defined on whole M. For example, consider the manifold $\mathbf{R}^3$ with metric

$$\mathrm{d}s^2 = e^{2\alpha(x)}\mathrm{d}x^2 + e^{2\beta(x)}(\mathrm{d}y^2+\mathrm{d}z^2), \tag{1}$$

where $\alpha(x)=\alpha(x+2\pi)$ and $\beta(x)=\beta(x+2\pi)$ are real periodic functions. This Riemannian manifold, denoted as $\tilde{M}_1$ hereafter, possesses three independent Killing vectors

$$\xi_2 = \frac{\partial}{\partial y},\ \xi_3 = \frac{\partial}{\partial z},\ \xi_4 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}. \tag{2}$$

Introducing identifications in $\tilde{M}_1$ by infinite actions generated by

$$\begin{aligned} g_1 \equiv e^{\pi\xi_4}e^{2\pi\xi_1} &: \ (x,y,z)\to(x+2\pi,-y,-z),\\ g_2 \equiv e^{2\pi\xi_2} &: \ (x,y,z)\to(x,y+2\pi,z),\\ g_3 \equiv e^{2\pi\xi_3} &: \ (x,y,z)\to(x,y,z+2\pi), \end{aligned} \tag{3}$$

where $\xi_1 \equiv \frac{\partial}{\partial x}$ is a vector field defined for convenience, we obtain a closed manifold M_1 homeomorphic to $T^3/\mathbf{Z}_2$. (Here, an exponential of a vector represents the diffeomorphism generated by the vector.) See Fig.1. Note that ξ_2 (and ξ_3) on the bottom points the opposite direction to that on the top, which fact tells us that ξ_2 and ξ_3 cannot be defined on the whole M_1. ξ_4 is also incompatible with the identifications by the translations by g_2 and g_3. Thus, M_1 admits no Killing vectors, though we can define Killing vectors on a neighborhood of every point.

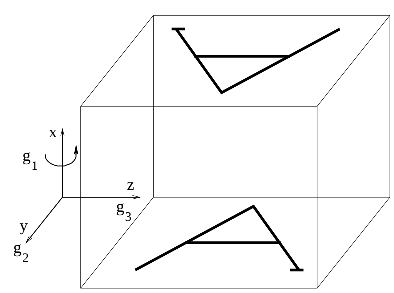

Figure 1: $T^3/\mathbf{Z}_2$, generated from $\mathbf{R}^3$ endowed with the Riemannian metric (1), by three identification generators g_1, g_2, and g_3. g_2 and g_3 are translations along, respectively, y and z-axes. g_1 is the composition of a translation along x-axis and the rotation in the y-z planes by π.

If we consider a spacetime manifold $(M_1\times\mathbf{R}, \mathrm{g}_{ab})$ whose spatial metric coincides with Eq.(1), this gives a simplest (but rather uninteresting) example of a generalized Gowdy model.

Definition Suppose a Riemannian manifold (M, h_{ab}) possesses the following two properties:

(i) There exists an open cover $\{O_i\}$ of M such that every $(O_i, h_{ab}^{(i)})$ admits m independent Killing vectors $\xi_1^{(i)} \cdots \xi_m^{(i)}$, where $h_{ab}^{(i)}$ is the natural metric on O_i inherited from h_{ab}.

(ii) On each $O_i \cap O_j (\neq \emptyset)$, $\xi^{(i)}$ and $\xi^{(j)}$ are related through a linear transformation

$$\xi_\alpha^{(i)} = \sum_{\beta=1}^{m} f_\alpha^{(ij)\beta} \xi_\beta^{(j)}, \quad \alpha = 1 \sim n, \tag{4}$$

where $f_\alpha^{(ij)\beta}$ is a nondegenerate constant matrix (for fixed i and j).

By *local Killing vectors on* (M, h_{ab}), we mean the collection of the pairs $(\{O_i\}, \{\xi_1^{(i)} \cdots \xi_m^{(i)}\})$ of the cover and the set of Killing vectors on each patch. For simplicity, we also simply denote them as $\xi_1 \cdots \xi_m$ if we do not have to say on which patch these Killing vectors are defined.

From the property (ii), the Killing *orbits* are well defined globally, even though the Killing vectors are local.

Definition A three-dimensional closed Riemannian manifold (M, h_{ab}) is called a *locally* $\mathrm{U}(1) \times \mathrm{U}(1)$ *symmetric space* (or an $\mathcal{LU}^2$*-symmetric space* in short) if (M, h_{ab}) admits two commuting local Killing vectors $\bigcup_i (\{O_i\}, \{\xi_1^{(i)}, \xi_2^{(i)}\})$, i.e., $[\xi_1^{(i)}, \xi_2^{(i)}] = 0$ on every O_i. We also impose the condition that this manifold must have no other local Killing vectors than those mentioned. We call this condition the *genericity condition.*

As we will see later, the Killing orbits for this kind of space are flat 2-tori if the local Killing vectors are nondegenerate. We can choose each patch O_i in the above definition as a regular neighborhood of such a T^2 (i.e., a neighborhood such that the boundaries coincides with another orbits), since the two commuting Killing vectors can be defined on such a neighborhood. In this case, the group $\mathrm{U}(1) \times \mathrm{U}(1)$ acts on this patch as isometries, hence the word "locally $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetric" for the whole manifold (M, h_{ab}). However, this terminology is rather lengthy to use frequently, so we also use "$\mathcal{LU}^2$-symmetric" or more simply "$\mathcal{LU}^2$-" in this paper.

Definition A spacetime manifold $(M \times \mathbf{R}, \mathrm{g}_{ab})$ is called a *locally* $\mathrm{U}(1) \times \mathrm{U}(1)$ *symmetric spacetime* (or a *generalized Gowdy spacetime*) if the spatial manifold M is closed and the spacetime admits two commuting spatial local Killing vectors. The genericity condition is understood, as in the case of the $\mathcal{LU}^2$-symmetric spaces.

Our first task is to determine all the possible topologies for the locally $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetric spaces. It is convenient to consider separately the case where the local Killing vectors degenerate on some points and the case where they do not on any points. We call the latter (nondegenerate) type of manifolds the *first kind*, and the former (degenerate) ones the *second kind.*

2.1 The first kind

Lemma 1 *If* (M, h_{ab}) *is an orientable locally* $\mathrm{U}(1) \times \mathrm{U}(1)$ *symmetric space of the first kind, then all Killing orbits must be closed and homeomorphic to* T^2*.*

Proof: Let (O, η_{ab}) be a Killing orbit, where η_{ab} is the induced metric on the orbit from h_{ab}. Since the Killing orbit possesses two commuting Killing vectors, η_{ab} is a flat metric. Since M is orientable, O is homeomorphic to one of $\mathbf{R}^2$, $\mathbf{R} \times S^1$, or T^2. First, let us assume $O \simeq \mathbf{R}^2$. Consider a geodesic l in (O, η_{ab}), and consider a point sequence $\{q_i\}$ such that all q_i are on l and q_i and q_{i+1} are unit distant with respect to η_{ab}. In M the sequence $\{q_i\}$ must have a converging subsequence $\{p_i\}$, since M is compact. Let V_ϵ be the neighborhood of the limit point p_∞ with radius ϵ, i.e., $|p - p_\infty| < \epsilon$ for all $p \in V_\epsilon$, where $|\cdot|$ is the geodesic distance with respect to h_{ab}. V_ϵ contains all p_i $(i > N_\epsilon)$ for an integer N_ϵ, so that for any p_i and p_j, $(i, j > N_\epsilon)$, $|p_i - p_j| < 2\epsilon$ as a result of the triangle inequality. On the other hand, the geodesic l is a "straight line" in the Euclid plane $(\mathbf{R}^2, \eta_{ab})$, so that the distant between any two points p_i and p_j $(i \neq j)$ in O is larger than or equal to unity. Hence we can choose a sequence of "isometric disconnected neighborhoods" $\{U_i\}$ in M such that $p_i \in U_i$, $S_i \cap S_j = \emptyset$, where $S_i \equiv U_i \cap O$, and there exist an isometry $\phi_{ij}:\ (p_i, S_i, U_i) \to (p_j, S_j, U_j)$ for all i, j. Since $S_i \cap S_j = \emptyset$ and $|p_i - p_j| < 2\epsilon$, we must have the consequence $p_j \notin S_i$ but $p_j \in U_i$ for a sufficiently small ϵ (i.e., for sufficiently large i and j). Moreover, this implies that for an *arbitrarily small* $\epsilon' < \epsilon$ there exist two points in M, p_i and p_j, such that $|p_i - p_j| < 2\epsilon'$, $p_j \notin S_i$ but $p_j \in U_i$, and there exist the isometry $\phi_{ij}:\ p_i \to p_j$. This in turn implies that there exist a third motion in (M, h_{ab}) off the orbit, which fact is against the genericity condition. (That is, $\mathbf{R}^2$-orbits can be formed only if (M, h_{ab}) is locally homogeneous.) In the case of $O \simeq \mathbf{R} \times S^1$, we can choose a geodesic l which is open as in the case of $O \simeq \mathbf{R}^2$. Repeating a similar argument we conclude that this case is also against the genericity. Thus, the only possibility is the case $O \simeq T^2$. □

Theorem 2 *If (M, h_{ab}) is an orientable locally $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetric space of the first kind, then M is homeomorphic to a T^2-bundle over S^1, for which the local Killing vectors generate the T^2-fibers.*

Proof: Consider a normal vector field that is everywhere nonvanishing and normal to the Killing orbit, and consider the integration curve c of this field passing through an arbitrarily given point $p \in M$. Let O be the Killing orbit to which p belongs, so c intersects with O at p. If c did not intersect again with O, then it would imply that the total volume of M is infinite, which fact contradicts with the compactness of M. Hence c intersects with O again. This implies that if we regard each Killing orbit as a point, then M reduces to S^1. Since all orbits are homeomorphic to T^2 as in Lemma 1, M is a T^2-bundle over S^1. □

Topologically, any T^2-bundle over S^1 can be obtained by first considering the product $T^2 \times I$, where $I = [0, 1]$ is the unit interval, then identifying $T^2 \times \{0\}$ and $T^2 \times \{1\}$ by the action of a gluing map $\phi:\ T^2 \to T^2$. Since any two gluing maps which are homotopic to each other results in topologically the same manifold, we usually think of the gluing map ϕ as an element of the *mapping class group* of T^2, $\mathrm{mcg}(T^2)$. Here, the mapping class group of a manifold M is the group of diffeomorphisms of M modulo diffeomorphisms which are homotopic to the identity, $\mathrm{mcg}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$. The group $\mathrm{mcg}(T^2)$ is well known as the *modular group*, $\mathrm{GL}(2, \mathbf{Z})$. However, since we are interested only in orientable manifolds, we consider the orientation-preserving mapping class group of the torus, $\mathrm{mcg}_+(T^2) \simeq \mathrm{SL}(2, \mathbf{Z})$. Letting

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}), \tag{5}$$

the fundamental group of the corresponding space is represented by

$$\pi_1 = \left\langle g_1, g_2, g_3;\ [g_2, g_3] = 1,\ g_1 g_2 g_1^{-1} = g_2^p g_3^r,\ g_1 g_3 g_1^{-1} = g_2^q g_3^s \right\rangle, \tag{6}$$

where g_2 and g_3 are generators of the fiber T^2 and g_1 is that of the base S^1. We therefore have a natural map $\omega_1 : \mathrm{SL}(2, \mathbf{Z}) \to W_1$, where W_1 is the set of orientable T^2-bundles over S^1. While ω_1 is not injective, we have the following useful theorem, of which proof can be found in Ref. [13] (Theorem 5.5 therein);

Theorem 3 ([13]) *Let M be the total space of a T^2-bundle over S^1 with gluing map ϕ, and let $A \in \mathrm{GL}(2, \mathbf{Z})$ represent the automorphism of $\pi_1(T^2)$ induced by ϕ. Then, M admits a Sol-structure if A is hyperbolic, an E^3-structure if A is periodic, or a Nil-structure otherwise. In particular, if $|\mathrm{Tr}A| > 2$, then A is hyperbolic, so M admits a Sol-structure.*

This theorem tells us that every T^2-bundle over S^1 admits a "geometric structure" [14], in other words, admits a locally homogeneous metric. This structure is, as in the above theorem, one of the three types, E^3, Nil, or Sol. Conversely, we can find all the T^2-bundles over S^1 from the closed quotients of E^3, Nil, and Sol, so we can refer to known classifications of the closed quotients of the three geometries to classify T^2-bundles over S^1. This procedure simultaneously determines the corresponding geometric structure for each representative. (To avoid confusions we remark that locally homogeneous metrics are used just for convenience here to discuss topologies, but we will see that appropriate inhomogeneous metrics are obtained by "relaxing" such a metric.)

The case of E^3: All the orientable closed manifolds modeled on E^3 are classified into six manifolds [15], T^3, $T^3/\mathbf{Z}_2$, $T^3/\mathbf{Z}_3$, $T^3/\mathbf{Z}_4$, $T^3/\mathbf{Z}_6$, and $T^3/\mathbf{Z}_2 \times \mathbf{Z}_2$. All of these *except* the last one can be regarded as T^2-bundle over S^1. Associated with these five are the representative elements $E_\lambda \in \mathrm{SL}(2, \mathbf{Z})$ $(\lambda = 1, 2, 3, 4, 6, \text{respectively})$:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ E_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{7}$$

The case of Nil: All the orientable closed manifolds modeled on Nil can also be completely classified (See page 4878 of Ref. [9]). Among them, ones which can be regarded as T^2-bundle over S^1 are given by the following two families $N_{\pm 1}(n) \in \mathrm{SL}(2, \mathbf{Z})$:

$$N_1(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ N_{-1}(n) = \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}, \tag{8}$$

where n is a nonzero integer. The parameter n in $N_1(n)$ can be chosen to be positive, since the relabeling $(g_1, g_3) \to (g_3, g_1)$ reverses the sign of n. For $N_{-1}(n)$, the opposite sign of n corresponds to a distinct topology.

The case of Sol: As in Theorem 3, if $|\mathrm{Tr}A| > 2$, we have a T^2-bundle over S^1 modeled on Sol. All those for $|\mathrm{Tr}A| \leq 2$ are non-orientable [13], so we will not consider them. Any two ones with distinct values of $\mathrm{Tr}A$ such that $|\mathrm{Tr}A| > 2$ are not homeomorphic, so one parameter family of $\mathrm{SL}(2, \mathbf{Z})$ [9],

$$S(n) = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}, \tag{9}$$

where n is an integer such that $|n| > 2$, gives a one parameter family of distinct T^2-bundles over S^1 modeled on Sol. However, this does *not* implies that the ones modeled on Sol can be classified only by $\mathrm{Tr}A$ like Eq.(9), since there exist topologically distinct manifolds with the same $\mathrm{Tr}A$ [16]. No complete classification of the closed manifolds modeled on Sol seems to be known. Nevertheless, as we will see in the next section, the sequence (9) gives nice representatives as reduced relativistic models.

2.2 The second kind

Theorem 4 *If (M, h_{ab}) is a locally* $\mathrm{U}(1) \times \mathrm{U}(1)$ *symmetric space of the second kind, then M is homeomorphic to one of S^3, $S^2 \times S^1$, or a lens space $L(p, q)$.*

Proof: Since M is closed, if removing an open neighborhood of the degenerate points from M, the resulting manifold $\hat{M}$ with boundaries is compact. Moreover, the boundaries can be chosen so as to coincide with (regular) Killing orbits. Let $\partial\hat{M}$ be the boundaries of $\hat{M}$ so chosen. Note that in the proof of Lemma 1, only the compactness of M is assumed to show that the Killing orbits are T^2. Hence all the Killing orbits for $(\hat{M}, \hat{h}_{ab})$ are also T^2. (Here, $\hat{h}_{ab}$ is the restriction of h_{ab} to $\hat{M}$.) In particular, every connected component of $\partial\hat{M}$ is T^2. Consider a (nonvanishing) normal vector field with respect to the Killing orbits in $(\hat{M}, \hat{h}_{ab})$. This flow of the normal vector field defines a unique one-to-one correspondence between boundary components, since the image of a boundary component by this flow must end up with another boundary component, due to the compactness of $\hat{M}$. Since M and therefore $\hat{M}$ are assumed to be connected, $\hat{M}$ must have only two boundary components and is naturally homeomorphic to the product $T^2 \times I$, where $I \equiv [0, 1]$ is the unit interval. Next, consider the neighborhood of a degenerate Killing orbit removed. An action of $\mathrm{U}(1) \times \mathrm{U}(1)$ can degenerate only to $\mathrm{U}(1)$, so a connected set of degenerate points forms a circle, and a neighborhood of such a circle with boundary forms a solid torus. The boundary, which is homeomorphic to T^2, can again be chosen so as to coincide with a (regular) Killing orbit. Consider two such neighborhoods, V_1 and V_2. Then, the original manifold is recovered by attaching them to $\hat{M}$ along the boundaries: $M \simeq V_1 \cup \hat{M} \cup V_2$. Note, however, that $V_1 \cup \hat{M}$ is another solid torus with the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry. Let us rewrite this manifold as V_1. The original manifold M is now obtained simply by identifying the boundaries of the two solid tori V_1 and V_2, $M \simeq V_1 \cup V_2$. Finally, it is a well-known fact (e.g. [17]) that the sum of two solid tori is homeomorphic to one of S^3, $S^2 \times S^1$, or a lens space $L(p, q)$. $\square$

Cosmological models based on $\mathcal{LU}^2$-symmetric spatial manifolds of the second kind can be thought of as a global generalization of cylindrically symmetric (plane wave) models. In fact, the symmetry axis of a cylindrically symmetric space is a degenerate orbit for the $\mathrm{U}(1) \times \mathrm{U}(1)$ action of the cylindrical symmetry. If, as usual, taking this axis as z-coordinate axis, and introducing identifications $z \sim z + z_0$ for a constant z_0, we have the symmetry axis that is a circle. A regular neighborhood of this circle is a solid torus with the $\mathrm{U}(1) \times \mathrm{U}(1)$ action. An $\mathcal{LU}^2$-manifold M can be obtained by identifying (the boundaries of) such two $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetric solid tori.

Now, it should be remarked that the local Killing vectors for an $\mathcal{LU}^2$-manifold of the second kind are always globally defined. This is because this kind of space can also be regarded as the product $T^2 \times I$ (with a metric that degenerates on the boundaries). The original Gowdy

models [2] in fact include all the models based on S^3, $S^2 \times S^1$, and the lens spaces $L(p,q)$. We therefore do not discuss much about these models in subsequent sections.

We end this subsection with a summary of the classification (e.g. [17]) of the lens spaces, for completeness. Consider a gluing map $\phi : T^2 \to T^2$ to identify the boundaries of two $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetric solid tori, ∂V_1 and ∂V_2. Since if two gluing maps are homotopic the resulting manifolds are homeomorphic, it is again natural to think of an identification map as an element of the mapping class group, $\phi \in \mathrm{mcg}(T^2) \simeq \mathrm{GL}(2, \mathbf{Z})$. Let $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be, respectively, meridian and longitudinal loops of ∂V_1 or ∂V_2. We define the action of $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ by the left action to these vectors. Then, the closed manifold M made by gluing the two solid tori with respect to A is the lens space $L(p,q)$. (The topology of M is regardless to r and s, i.e., it is determined only by the mapping of the meridian loop of ∂V_1.) The integers p and q are coprime and we can set $p \geq 0$ (since $L(p,q) \simeq L(-p,q)$). S^3 and $S^2 \times S^1$ correspond, respectively, to $L(1,q)$ and $L(0,1)$. Lens spaces are completely classified by the fact that $L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $p = p'$, and $q \equiv \pm q' \mod p$ or $qq' \equiv \pm 1 \mod p$.

3 Relaxation, Reduction and Dynamical Equivalence

In this section we present an detailed account of the reduction procedure for the $\mathcal{LU}^2$-symmetric models of the *first kind*, with emphasis on the significance of the corresponding geometric structures. We will also obtain useful results, the "dynamical equivalences".

First, we remark that since the spacetime manifold is the product $M \times \mathbf{R}$, where the spatial manifold M is a (T^2-)bundle over S^1 with fibers generated by the local Killing vectors, the natural reduced manifold obtained by contracting the Killing orbits should be $S^1 \times \mathbf{R}$. However, as we see bellow, when the bundle is not (covered by) a direct product, this reduction is not trivial.

We represent an $\mathcal{LU}^2$-symmetric spacetime (of the first kind) $(M \times \mathbf{R}, \mathrm{g}_{ab})$ with the covering map

$$\pi : \ (\tilde{M} \times \mathbf{R}, \tilde{\mathrm{g}}_{ab}) \to (M \times \mathbf{R}, \mathrm{g}_{ab}) = (\tilde{M} \times \mathbf{R}, \tilde{\mathrm{g}}_{ab})/\Gamma, \tag{10}$$

where $(\tilde{M} \times \mathbf{R}, \tilde{\mathrm{g}}_{ab})$ is the universal covering manifold of $(M \times \mathbf{R}, \mathrm{g}_{ab})$, and Γ is a covering group, which acts on $\tilde{M}$ discretely. Since the spatial manifold M is a T^2-bundle over S^1, $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$. We can therefore use globally defined coordinates, say $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$, to represent $\tilde{\mathrm{g}}_{ab}$. A natural form of the metric admitting two commuting Killing vectors represented with these coordinates is $\mathrm{g}_{\mu\nu}(t,x)dx^\mu dx^\nu$, where Greek indices $\mu, \nu, \cdots$ run over 0 to 3. Moreover, fixing the freedom of diffeomorphisms in $\mathrm{g}_{\mu\nu}(t,x)dx^\mu dx^\nu$ as much as possible, we can represent the metric $\tilde{\mathrm{g}}_{ab}$ as [2]

$$\mathrm{d}s^2 = e^{\gamma/2}(-\mathrm{d}t^2 + \mathrm{d}x^2) + 2N_A \mathrm{d}x^A \mathrm{d}t + Re_{AB}\mathrm{d}x^A \mathrm{d}x^B, \tag{11}$$

where γ, R, N_A and e_{AB} (Capital indices $A, B, \cdots$ run 2 to 3) are functions of t and x. We set $\det e_{AB} = 1$ so that R describes the area of the Killing orbit. This metric is called *the generic*

metric of the canonical form. We will also consider the restricted metric with $N_A = 0$ (the "two-surface orthogonality" [2])

$$ds^2 = e^{\gamma/2}(-dt^2 + dx^2) + Re_{AB}dx^A dx^B, \tag{12}$$

which is called *the two-surface orthogonal metric of the canonical form.* For a standard prescription of Einstein's equations for the last metric, see Appendix A. (Note that the metric functions are represented with "bars" in this Appendix. We will take this notation from the next subsection to avoid conflicts with "noncanonical" metrics which will appear.)

Both metrics admit two dimensional isometry group consisting of translations along y and z axes. Note that the y-z planes descend to the T^2-fibers after appropriate identifications in each y-z plane. These identifications are generated by two independent vectors, which we can choose without loss of generality as the unit coordinate generators, $\partial/\partial y$ and $\partial/\partial z$. Then we can naturally think of the actions of these generators as the representation of the generators, g_2 and g_3, of π_1 into the isometry group; $g_2 \simeq e^{\frac{\partial}{\partial y}}$, $g_3 \simeq e^{\frac{\partial}{\partial z}}$. The remaining generator g_1 must induce a translation along x-axis *plus* the modular transformation induced by an element $A = n^A{}_B \in \mathrm{SL}(2, \mathbf{Z})$. Thus, $g_1 : \ (x, y, z) \to (x + 2\pi, n^A{}_B x^B)$ on the t =constant space. Together with $g_3 : (x, y, z) \to (x, y + 2\pi, z)$ and $g_3 : (x, y, z) \to (x, y, z + 2\pi)$, all the relations in the fundamental group (6) are fulfilled. The boundary conditions for the metric (11) or (12) is determined by the requirement that the action of g_1 be an isometry of the metric, which can be easily found;

$$\begin{aligned} &\lambda(t, x) = \lambda(t, x + 2\pi),\ R(t, x) = R(t, x + 2\pi), \\ &N_A(t, x) = n^B{}_A N_B(t, x + 2\pi),\ e_{AB}(t, x) = n^C{}_A n^D{}_B e_{CD}(t, x + 2\pi). \end{aligned} \tag{13}$$

Note that the components e_{AB} that describe each T^2-fiber are *not* periodic functions in general, so that the reduced manifold (spanned by t and x) cannot be naturally regarded as $S^1 \times \mathbf{R}$, if we represent the metric as Eq.(11) or (12). The reason why e_{AB} do not automatically become periodic is that what two independent components in e_{AB} themselves describe are Teichmüller parameters for the T^2-fiber, rather than moduli parameters. (For the difference between Teichmüller and moduli parameters, see, e.g., Ref. [14].)

An idea to remedy the situation is to choose the metric functions so that they become constant when the spatial metric is at a locally homogeneous limit. That is, conversely, we "relax" a locally homogeneous metric in an appropriate way to obtain a suitable metric. Note that a constant function is naturally a function on S^1. As far as considering smooth deformations of the metric, metric functions chosen in such a way must continue to be (smooth) functions on S^1 even when the metric becomes inhomogeneous (with the $\mathcal{LU}^2$ symmetry).

The scheme One nice way to realize this idea is to expand the spatial metric in terms of the (left) invariant one-forms σ^i ($i = 1 \sim 3$) of a Bianchi group G. As we have seen in Sec.2.1, any T^2-bundle over S^1 admits one of E^3, Nil, or Sol-structure. These geometric structures correspond, respectively, to Bianchi I(VII$_0$), II, and VI$_0$. (E^3 has multiple correspondences.) The appropriate Bianchi group G is determined from this correspondence. Let ξ_i ($i = 1 \sim 3$) be independent generators of G, that is, ξ_i are Killing vectors of a G-invariant metric. Since σ^i are by definition invariant under the action of G, the homogeneous metric is written as

$h_{ij}\sigma^i\sigma^j$ with (spatially) constant components h_{ij}. The Killing vectors of this homogeneous metric include a commuting pair. Suppose $[\xi_2, \xi_3] = 0$. Let χ be a vector field such that χ is independent of ξ_2 and ξ_3 at every point and is invariant under the action generated by ξ_2 and ξ_3, i.e., $\mathcal{L}_{\xi_2}\chi = \mathcal{L}_{\xi_3}\chi = 0$. Here, $\mathcal{L}$ represents the Lie derivative. Since $[\xi_2, \xi_3] = 0$, we can choose coordinates (y and z) generated by ξ_2 and ξ_3. Moreover, since $\mathcal{L}_{\xi_2}\chi = \mathcal{L}_{\xi_3}\chi = 0$ implies the commutativity $[\xi_2, \chi] = [\xi_3, \chi] = 0$, we can choose a third coordinate (x) as that of generated by χ. Consider an arbitrary function $f(x)$ that depends only on x chosen in this way. Then, the level set of $f(x)$ is invariant under the diffeomorphisms generated by ξ_2 and ξ_3, and it coincides with the set of the orbits generated by ξ_2 and ξ_3. Hence we can write the inhomogeneous metric as $h_{ij}(x)\sigma^i\sigma^j$, which is invariant under the diffeomorphisms generated by ξ_2 and ξ_3, and inhomogeneous in the desired manner. (Recall that all σ^i are invariant under ξ_2 and ξ_3.) The metric functions $h_{ij}(x)$ should be periodic in x because of the reason explained above. Hence, a spacetime metric with $h_{ij}(t, x)\sigma^i\sigma^j$ as the spatial part naturally gives rise to a reduction onto $S^1 \times \mathbf{R}$. We call this scheme the *relaxation method.* If $\xi_1 = \chi$ as in the Bianchi I case, then this scheme descends to the usual coordinate representation like Eq.(11) or (12), but otherwise it is nontrivial. $\square$

Recall that a Bianchi group G is a three-dimensional simply transitive group acting on a three-dimensional simply connected manifold $\tilde{M}$, and therefore we can identify G with $\tilde{M}$ (e.g. [12]). We adhere to this viewpoint in this paper and use the components of G to represent the coordinates of $\tilde{M}$, as well.

One comment should be made here. As pointed out in Ref. [3], all Bianchi homogeneous spaces except for VIII and IX possess a commuting pair of Killing vectors. Even for VIII and IX, if considering higher symmetry there can exist commuting Killing vectors. Moreover, these homogeneous spaces can be compactified to closed spaces, except for IV and VI$_a$ types. Therefore all the locally homogeneous manifolds of the Bianchi types except IV and VI$_a$ seems to be locally homogeneous limits of $\mathcal{LU}^2$-manifolds. However, as we mentioned above, only Bianchi I, VII$_0$, II, and VI$_0$ types can be such a limit for the first kind manifolds. The second kind manifolds correspond to Bianchi IX (in case of $M \simeq S^3$ and $L(p, q)$) and the Nariai-Kantowski-Sachs type (in case of $M \simeq S^2 \times S^1$). Thus, *Bianchi III, V, VII$_a$, VIII types are missing in the list of possible manifolds as the locally homogeneous limits of our $\mathcal{LU}^2$-manifolds.* The reason is because each orbit of the two commuting local Killing vectors for a locally homogeneous manifold of these types always do not close. For example, in the case of Bianchi V, it forms $\mathbf{R}^2$. As we have seen in the proof of Lemma 1, such a case is possible only when the manifold is locally homogeneous. That is, we can say that, as a corollary to Theorems 2 and 4:

Corollary 5 *Every closed locally homogeneous manifold of Bianchi III, V, VII$_a$ and VIII types cannot be relaxed to be inhomogeneous with $\mathcal{LU}^2$ symmetry.*

In this sense, these manifolds are "stiff." It is worth noting that a common feature of them is that they all contain a hyperbolic structure, H^2 or H^3.

In the following subsections, we perform the reduction procedure for each type of Nil, Sol, and E^3. The spacetime manifold $M \times \mathbf{R}$ reduces to $S^1 \times \mathbf{R}$ for any spatial manifold M. One important feature that is found as a result of the reductions is that, even for models with topologically distinct spatial manifolds, distinctions at the reduced level can completely *degenerate.* In other

words, we obtain for some models the same set of PDEs (as the reduced Einstein equations) with the same boundary conditions. For such models the universal covering spacetime metrics can be represented in exactly the same form with only the difference contained in the covering group Γ. We call that such models are *dynamically equivalent* to each other.

3.1 Case of Nil

The Bianchi II group $G_{\rm II}$ is the three-dimensional simply transitive group with the multiplication rule

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z+ay \end{pmatrix}. \tag{14}$$

This group is generated by

$$\xi_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \xi_2 = \frac{\partial}{\partial y},\ \xi_3 = \frac{\partial}{\partial z}. \tag{15}$$

The one-forms which are invariant under the action of $G_{\rm II}$ are

$$\sigma^1 = \mathrm{d}x,\ \sigma^2 = \mathrm{d}y,\ \sigma^3 = \mathrm{d}z - x\mathrm{d}y. \tag{16}$$

Since ξ_2 and ξ_3 commute and they generate coordinates y and z, a desired representation of the spatial metric is given by $h_{ij}(x)\sigma^i\sigma^j$.

Before proceeding further, however, we have to clarify a subtle point about the fact that we can choose another commuting pairs of group generators. For example, ξ_1 and ξ_3 also commute to each other, and the orbits they generate are distinct from those generated by ξ_2 and ξ_3. In general, since the commutation rule for linear combinations of the group generators is given by

$$[\alpha^i\xi_i, \beta^j\xi_j] = \alpha^i\beta^j[\xi_i, \xi_j] = -(\alpha^1\beta^2 - \alpha^2\beta^1)\xi_3, \tag{17}$$

any two generators $\alpha = \alpha^i\xi_i$ and $\beta = \beta^j\xi_j$ commute if and only if $\alpha^1\beta^2 - \alpha^2\beta^1 = 0$. The last condition implies that ξ_3 must be tangent to the surface spanned by α and β if α and β are independent. Such a surface, conversely, is spanned by $\eta_2 \equiv \sin\theta\xi_1 + \cos\theta\xi_2$ and $\eta_3 \equiv \xi_3$, where θ is a real parameter. Hence, we have one-parameter (θ) family of distinct sets of orbits generated by a commuting pair. This then implies that we have one-parameter degree of freedom of relaxing a locally homogeneous manifold of Bianchi II type.

However, this fact is actually insignificant if viewing the metric as a universal cover metric. That is, the freedom of θ can be absorbed in the freedom of choosing the covering group Γ. Therefore we do not have to consider the freedom of θ, which can be fixed $\theta = 0$ without loss of generality. (The model based on $\theta = \pi/2$ was presented in Ref. [3] as 'Type 2', which is redundant for this reason.)

Now, we have established the fact that the relaxed (universal covering) metric can be represented by $h_{ij}(x)\sigma^i\sigma^j$ with the basis (16). The form of spacetime metric corresponding to the two-surface orthogonal class of metrics (85) is given by

$$\mathrm{d}s^2 = e^{\gamma/2}(-\mathrm{d}t^2 + (\sigma^1)^2) + R[e^P(\sigma^3 + Q\sigma^2)^2 + e^{-P}(\sigma^2)^2], \tag{18}$$

where the metric functions γ, R, P and Q are functions of t and x, and periodic with respect to x. (We have changed the choice of P and Q from that in Ref. [3], since the present choice gives us the most natural and simplest correspondence to $\bar{P}$ and $\bar{Q}$. See bellow.) These metric functions are related to those for the canonical representation (85) through

$$\bar{\gamma} = \gamma,\ \bar{R} = R,\ \bar{P} = P,\ \bar{Q} = Q - x. \tag{19}$$

The reduced Einstein equations for the unbarred variables γ, P, Q, and R are therefore easily obtained by substituting these equations into those for the canonical (i.e. barred) variables presented in Appendix A.

Our next task is to check that we can indeed compactify the universal cover. We choose the period for the metric functions along the x axis as 2π:

$$f(t,x) = f(t, x+2\pi), \quad \text{for } f = \gamma,\ R,\ P \text{ and } Q. \tag{20}$$

We denote the isometry group for the spacetime metric (18) as $H_{\rm II}^{\langle 2\rangle}$, and its subgroup which is a subgroup of $G_{\rm II}$ as $H_{\rm II} \subset G_{\rm II}$. We have

$$H_{\rm II} = \left\{ \left(\begin{array}{c} 2\pi n \\ b \\ c \end{array} \right) \in G_{\rm II} \middle| n \in \mathbf{Z}, b, c \in \mathbf{R} \right\}. \tag{21}$$

The full isometry group $H_{\rm II}^{\langle 2\rangle}$ is generated by $H_{\rm II}$ and the $\mathbf{Z}_2$-isometry

$$k : (x,y,z) \to (x,-y,-z). \tag{22}$$

Recall that the fundamental groups for $N_1(n)$ and $N_{-1}(n)$ are given by Eqs.(8) with (6). Putting

$$g_i = \left(\begin{array}{c} 2\pi n_i \\ {g_i}^2 \\ {g_i}^3 \end{array} \right), \quad i = 1 \sim 3, \tag{23}$$

we have to solve the relations in these fundamental groups for the parameters ${g_i}^j$ and n_i. The solutions are easily found for $N_1(n)$. For $N_{-1}(n)$ we have to think of g_1 as a composite of k and an element of $H_{\rm II}$. The solutions are given by
For $N_1(n)$:

$$\begin{aligned} \Gamma_n &= \{g_1, g_2, g_3\} \\ &= \left\{ \left(\begin{array}{c} 2\pi p \\ {g_1}^2 \\ {g_1}^3 \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \\ \frac{2\pi}{n}(p{g_3}^2 - q{g_1}^2) \end{array} \right), \left(\begin{array}{c} 2\pi q \\ {g_3}^2 \\ {g_3}^3 \end{array} \right) \right\}, \end{aligned} \tag{24}$$

where $p, q \in \mathbf{Z}$, ${g_i}^j \in \mathbf{R}$, and $p{g_3}^2 - q{g_1}^2 \neq 0$;
For $N_{-1}(n)$:

$$\begin{aligned} \Gamma_n &= \{g_1, g_2, g_3\} \\ &= \left\{ k \circ \left(\begin{array}{c} 2\pi p \\ {g_1}^2 \\ {g_1}^3 \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \\ -\frac{2\pi p {g_3}^2}{n} \end{array} \right), \left(\begin{array}{c} 0 \\ {g_3}^2 \\ {g_3}^3 \end{array} \right) \right\}, \end{aligned} \tag{25}$$

where $p \in \mathbf{Z}$, $g_i{}^j \in \mathbf{R}$, and $pg_3{}^2 \neq 0$. The point here is that there exist solutions, ensuring that we can compactify the universal cover for any closed spatial manifold $N_{\pm 1}(n)$. For completeness we remark that the parameters appearing in Γ_n can be fixed arbitrarily, since redefinitions of metric functions can make the parameters equal to arbitrary values. For example, we can take $p = g_3{}^2 = 1$, and $q = g_1{}^2 = g_1{}^3 = g_3{}^3 = 0$ for $N_1(n)$, and $p = g_3{}^2 = 1$, and $g_1{}^2 = g_1{}^3 = g_3{}^3 = 0$ for $N_{-1}(n)$.

Note that we did not have to impose any other condition than Eq.(20) for all spatial topologies $N_{\pm 1}(n)$. This results in obtaining the same reduced Einstein equations with the same boundary conditions (20). Thus:

Proposition 6 *All (two-surface orthogonal) $\mathcal{LU}^2$-symmetric models of Nil type are dynamically equivalent.*

Here, the two-surface orthogonality is necessary for $N_{-1}(n)$ models to ensure that the map k is an isometry. That is, generic $N_{-1}(n)$ models with shift functions are not allowed. On the other hand, all generic $N_1(n)$ models are dynamically equivalent.

3.2 Case of Sol

The invariant one-forms are given by

$$\sigma^1 = \mathrm{d}x,\ \sigma^2 = \frac{1}{\sqrt{2}}(e^{\mathrm{q}x}\mathrm{d}y + e^{-\mathrm{q}x}\mathrm{d}z),\ \sigma^3 = \frac{1}{\sqrt{2}}(-e^{\mathrm{q}x}\mathrm{d}y + e^{-\mathrm{q}x}\mathrm{d}z), \tag{26}$$

where $\mathrm{q} > 0$ is a positive parameter introduced for convenience.

Using these 1-forms we can write the two-surface orthogonal metric as

$$\mathrm{d}s^2 = e^{\gamma/2}(-\mathrm{d}t^2 + (\sigma^1)^2) + R[e^P(\sigma^{2\prime} + Q\sigma^{3\prime})^2 + e^{-P}(\sigma^{3\prime})^2], \tag{27}$$

where γ, P, Q, and R, are functions of t and x, and they are assumed to be periodic in x. We have defined

$$\sigma^{2\prime} \equiv \frac{1}{\sqrt{2}}(\sigma^2 - \sigma^3) = e^{\mathrm{q}x}\mathrm{d}y, \quad \sigma^{3\prime} \equiv \frac{1}{\sqrt{2}}(\sigma^2 + \sigma^3) = e^{-\mathrm{q}x}\mathrm{d}z. \tag{28}$$

(The choice of metric functions P and Q here is different from that in Ref. [3]. We have done so for apparent simplicity, but we should keep in mind that at the homogeneous limit the metric is not diagonalizable with respect to $(\sigma^1, \sigma^{2\prime}, \sigma^{3\prime})$, in contrast to with respect to $(\sigma^1, \sigma^2, \sigma^3)$, for vacuum spacetimes.) The metric functions defined with Eq.(27) are related to those in the canonical representation (85) through

$$\bar{\gamma} = \gamma,\ \bar{R} = R,\ \bar{P} = P + 2\mathrm{q}x,\ \bar{Q} = e^{-2\mathrm{q}x}Q. \tag{29}$$

The reduced Einstein equations for γ, P, Q, and R can be obtained by substituting these equations into those presented in Appendix A.

For compactifications, see Appendix B. In particular, the isometry group of the metric (27) is given by the $H^{\langle 2\rangle}_{\mathrm{VI}_0}$ presented in the Appendix. Therefore we can compactify the universal cover possessing this metric for all A. If we choose the period along x-axis for the metric functions as

2π, we must put $c_3 = 2\pi$. Accordingly, we must choose the parameter q so that $e^{2\pi \mathrm{q}}$ equals to the greater absolute eigenvalue of the matrix A, since we have assumed $\mathrm{q} > 0$. Note that the characteristic polynomial (107) depends only on TrA. Let $n \equiv \mathrm{Tr}A$. Then we obtain

$$\mathrm{q} = \frac{1}{2\pi} \log \frac{|n| + \sqrt{n^2 - 4}}{2}. \tag{30}$$

The reduced Einstein equations for the (unbarred) metric functions depends (only) on the parameter q, and the boundary conditions are the common simple periodic boundary condition. Moreover, q depends only on $|n|$. We thus conclude:

Proposition 7 *An $\mathcal{LU}^2$-symmetric model of Sol type is specified with an element $A \in \mathrm{mcg}_+(T^2) \simeq \mathrm{SL}(2, \mathbf{Z})$ such that $|\mathrm{Tr}A| > 2$. Let A_1 and A_2 be such matrices. Then, the corresponding models are dynamically equivalent if $|\mathrm{Tr}A_1| = |\mathrm{Tr}A_2|$.*

From this fact, we find it suffices to consider the models with the one-parameter family $S(n)$ $(n > 2)$, defined in Eq.(9), as representatives.

3.3 Case of E^3

The corresponding Bianchi types are Bianchi I and VII$_0$. More precisely, geometry E^3 is the Euclid space $\mathbf{R}^3$ with the standard metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The isometry group of E^3 is therefore formed by the translations $\mathbf{R}^3$ and rotations O(3), and is isomorphic to the Poincaré group, Isom$E^3 \simeq$ IO(3). IO(3) contains two simply transitive subgroups, the Bianchi I ($G_{\mathrm{I}} \simeq \mathbf{R}^3$) and VII$_0$ (G_{VII_0}) groups, so that the correspondence above is arrived at.

We can make the inhomogeneous metrics from both G_{I} and G_{VII_0}. However, the effects are not equivalent. First, consider the metric made from G_{I}. Since the invariant one-forms coincide with the usual coordinate basis dx, dy, and dz, the spacetime metric and the boundary conditions are the same as the canonical ones (12) and (13). We find in particular that the metric functions for E_1 and E_2 are simply periodic. For E_3, E_4, and E_6, however, the metric functions must obey the last condition in Eqs.(13), which does not imply the simple periodic conditions, (though they have period 6π, 4π, and 6π for E_3, E_4, and E_6, respectively). As long as we adhere to the metric obtained from G_{I}, this feature is inevitable, since the closed manifolds corresponding to E_3, E_4, and E_6 cannot be realized in the Bianchi I group G_{I}, or $G_{\mathrm{I}}^{\langle 2\rangle}$. Here, $G_{\mathrm{I}}^{\langle 2\rangle}$ is generated by G_{I} and the $\mathbf{Z}_2$-map $h:\ (x, y, z) \to (x, -y, -z)$, which is an isometry of the metric (12). Precisely, the fundamental group $\pi_1(E_1)$ corresponding to E_1 can be embedded in G_{I}, and $\pi_1(E_2)$ can be embedded in the $G_{\mathrm{I}}^{\langle 2\rangle}$. However, other fundamental groups can be embedded neither in G_{I} nor $G_{\mathrm{I}}^{\langle 2\rangle}$. Since the isometry group of the metric (12) becomes a subgroup of $G_{\mathrm{I}}^{\langle 2\rangle}$, provided that the metric functions are all periodic with period 2π, this implies that the embeddings of π_1 for E_3, E_4, and E_6 into the subgroup are impossible. Thus, the appropriate boundary conditions cannot be the simple periodic conditions for them.

On the other hand, all the fundamental groups $\pi_1(E_\lambda)$ $(\lambda = 1, 2, 3, 4, 6)$ can be embedded into G_{VII_0} [9], which fact implies that the boundary conditions can become simple periodic ones for the metric obtained from G_{VII_0} for all E_λ. We present an explicit prescription bellow.

The invariant one-forms for Bianchi VII_0 are given by

$$\sigma^1 = \mathrm{d}x,\ \sigma^2 = \cos \mathrm{q}x\,\mathrm{d}y + \sin \mathrm{q}x\,\mathrm{d}z,\ \sigma^3 = -\sin \mathrm{q}x\,\mathrm{d}y + \cos \mathrm{q}x\,\mathrm{d}z, \tag{31}$$

where $\mathrm{q} > 0$ is a positive parameter. Using these 1-forms we can write the two-surface orthogonal metric as

$$\mathrm{d}s^2 = e^{\gamma/2}(-\mathrm{d}t^2 + (\sigma^1)^2) + R[e^P(\sigma^2 + Q\sigma^3)^2 + e^{-P}(\sigma^3)^2], \tag{32}$$

where γ, P, Q, and R, are functions of t and x, and they are assumed to be periodic in x with period 2π. The parameter q will be chosen so that we can compactify the universal cover. (The role of q is similar to the q appearing in the Sol model.) The correspondence to the barred metric functions in the canonical metric (85) is given by

$$\begin{aligned} e^{\bar{P}(x)} &= \Phi(P(x), Q(x), \cos \mathrm{q}x, \sin \mathrm{q}x), \quad \bar{Q}(x) = \Psi(P(x), Q(x), \cos \mathrm{q}x, \sin \mathrm{q}x), \\ e^{P(x)} &= \Phi(\bar{P}(x), \bar{Q}(x), \cos \mathrm{q}x, -\sin \mathrm{q}x), \quad Q(x) = \Psi(\bar{P}(x), \bar{Q}(x), \cos \mathrm{q}x, -\sin \mathrm{q}x). \end{aligned} \tag{33}$$

where the two functions Φ and Ψ are defined by, for arbitrary P, Q, c, and s,

$$\begin{aligned} \Phi(P, Q, c, s) &= e^P(c - sQ)^2 + e^{-P}s^2, \\ \Psi(P, Q, c, s) &= \frac{e^P(c - sQ)(s + cQ) - e^{-P}cs}{\Phi(P, Q, c, s)}. \end{aligned} \tag{34}$$

Note that the vacuum Einstein equations in terms of the unbarred variables contain the parameter q.

As usual, we represent the elements of the Bianchi group in the column vector form. The multiplication rule for G_{VII_0} is given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathrm{q}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{q}} = \begin{pmatrix} a + x \\ \begin{pmatrix} b \\ c \end{pmatrix} + R_{\mathrm{q}a} \begin{pmatrix} y \\ z \end{pmatrix} \end{pmatrix}_{\mathrm{q}}, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathrm{q}}^{-1} = \begin{pmatrix} -a \\ -R_{-\mathrm{q}a} \begin{pmatrix} b \\ c \end{pmatrix} \end{pmatrix}_{\mathrm{q}}, \tag{35}$$

where $R_{\mathrm{q}a}$ is the rotation matrix by angle $\mathrm{q}a$;

$$R_{\mathrm{q}a} = \begin{pmatrix} \cos \mathrm{q}a & -\sin \mathrm{q}a \\ \sin \mathrm{q}a & \cos \mathrm{q}a \end{pmatrix}. \tag{36}$$

The subscripts q appearing in the column vectors are to remind that the multiplication rule is defined with respect to q. The one-forms (31) are invariant under this action.

We denote the isometry group of the metric (32) which is a subgroup of G_{VII_0} as H_{VII_0}. It is given by

$$H_{\mathrm{VII}_0} = \left\{ \begin{pmatrix} 2\pi n \\ b \\ c \end{pmatrix}_{\mathrm{q}} \in G_{\mathrm{VII}_0} \middle|\, n \in \mathbf{Z},\ b, c \in \mathbf{R} \right\}. \tag{37}$$

The (full) isometry group $H^{\langle 2\rangle}_{\mathrm{VII}_0}$ of the metric (32) is generated by H_{VII_0} and the $\mathbf{Z}_2$-isometry $h:\ (x, y, z) \to (x, -y, -z)$.

We can perform the embedding of every $\pi_1(E_\lambda)$ both with and without the $\mathbf{Z}_2$-isometry h, as we can read from Sec.V A of Ref. [9]. (But, be careful with the differences in the choice of representations of the relations in π_1.) When we do not use h, i.e., embed the fundamental group $\pi_1(E_\lambda)$ into H_{VII_0}, the appropriate value of q is simply given by

$$\mathrm{q} = \frac{1}{\lambda}. \tag{38}$$

Hence all the models seem to be dynamically inequivalent. However, we can choose $\mathrm{q} = 1$ for $\lambda = 2$, and $\mathrm{q} = 1/3$ for $\lambda = 6$, if we use h. For example, in the case of E_6 one can confirm that the solution of the embedding into H_{VII_0} is given by

$$\begin{aligned}\Gamma_6 &= \{g_1, g_2, g_3\} \\ &= \left\{ \begin{pmatrix} 2\pi \\ {g_1}^2 \\ {g_1}^3 \end{pmatrix}_{\frac{1}{6}}, \begin{pmatrix} 0 \\ {g_2}^2 \\ {g_2}^3 \end{pmatrix}_{\frac{1}{6}}, \begin{pmatrix} 0 \\ -R_{\pm\pi/3} \begin{pmatrix} {g_2}^2 \\ {g_2}^3 \end{pmatrix} \end{pmatrix}_{\frac{1}{6}} \right\}\end{aligned} \tag{39}$$

with $\mathrm{q} = 1/6$. Here, ${g_i}^j$ are real parameters. Also, we can embed the fundamental group into $H^{\langle 2\rangle}_{\mathrm{VII}_0}$ as

$$\Gamma_{6'} = \left\{ h \circ \begin{pmatrix} 2\pi \\ {g_1}^2 \\ {g_1}^3 \end{pmatrix}_{\frac{1}{3}}, \begin{pmatrix} 0 \\ {g_2}^2 \\ {g_2}^3 \end{pmatrix}_{\frac{1}{3}}, \begin{pmatrix} 0 \\ R_{2\pi/3} \begin{pmatrix} {g_2}^2 \\ {g_2}^3 \end{pmatrix} \end{pmatrix}_{\frac{1}{3}} \right\} \tag{40}$$

with $\mathrm{q} = 1/3$. Since the fundamental group of E_3 can be embedded into H_{VII_0} with $\mathrm{q} = 3$ as

$$\Gamma_3 = \left\{ \begin{pmatrix} 2\pi \\ {g_1}^2 \\ {g_1}^3 \end{pmatrix}_{\frac{1}{3}}, \begin{pmatrix} 0 \\ {g_2}^2 \\ {g_2}^3 \end{pmatrix}_{\frac{1}{3}}, \begin{pmatrix} 0 \\ -R_{\pm 2\pi/3} \begin{pmatrix} {g_2}^2 \\ {g_2}^3 \end{pmatrix} \end{pmatrix}_{\frac{1}{3}} \right\}, \tag{41}$$

the E_6 and E_3 models are dynamically equivalent. (The significance of the parameters ${g_i}^j$ is the same as in the case of Nil. In particular, we can take ${g_1}^2 = {g_1}^3 = 0$, ${g_2}^2 = 1$, and ${g_2}^3 = 0$, in all cases above.)

As a result, we have only three dynamically inequivalent classes;

Proposition 8 *The E_1, E_3, and E_4 models comprise a set of representatives for the dynamical equivalence in the two-surface orthogonal $\mathcal{LU}^2$-symmetric models of E^3 type.*

For the generic models of E^3 type with the shift functions, all the E_λ models are dynamically inequivalent, since the map h is not an isometry.

To end this subsection, we remark again that the $\mathcal{LU}^2$-symmetric models of E^3 type best match with Bianchi VII_0, not Bianchi I. However, if limited to the T^3 (or $T^3/\mathbf{Z}_2$) model, which corresponds to E_1 (E_2), what it naturally corresponds is Bianchi I. To make later discussions easier, it may therefore be convenient to put forward the following convention.

Convention When speaking of the "E^3 models", we understand them to have correspondence to Bianchi VII_0. In particular, the "unbarred variables" for an E^3 model are those in the metric (32). However, when speaking of the "T^3 model", we understand it to have correspondence to Bianchi I. Therefore the unbarred variables coincide with barred variables appearing in the canonical metric (85).

4 Translation and Reflection symmetries

The Gowdy equations (92) have several symmetries. First of all, from the fact that these equations do not have explicit x-dependence, they have the natural translation symmetry with respect to $x \to x - a$, where a is a real parameter. This means that if $(\bar{P}(t,x), \bar{Q}(t,x))$ is a solution for the equations, then $(\bar{P}(t,x+a), \bar{Q}(t,x+a))$ is also a solution. One more symmetry that naturally comes to our attention is the reflection symmetry with respect to $x \to -x$, which is a result of the invariance of the equations under the transformation $\partial/\partial x \to -\partial/\partial x$. This symmetry means that if $(\bar{P}(t,x), \bar{Q}(t,x))$ is a solution for the equations, then $(\bar{P}(t,-x), \bar{Q}(t,-x))$ is also a solution. Another feature that results from these symmetries is that (in the Hamiltonian picture) an initial data that is symmetric with respect to $x \to x + a$ or $x \to -x$ maintains the same symmetry under the time evolution. This point will be discussed in the second subsection bellow.

For the usual Gowdy T^3 model, both $x \to x - a$ and $x \to -x$ transformations acting on $(\bar{P}(t,x), \bar{Q}(t,x))$ preserve the boundary conditions on $\bar{P}$ and $\bar{Q}$, so that they express true symmetries. For Nil and Sol models, however, these transformations do not preserve the boundary conditions, so that they do not express true symmetries. Nevertheless, we can find translation symmetries for Nil and Sol models in a generalized sense, and also find (generalized) reflection symmetries for them. We also find that the T^3 model admits larger amounts of reflection symmetries than mentioned above. In the first subsection bellow, we obtain these translation and reflection symmetries in a systematic way. Based on this result, in the second subsection we discuss the dynamical point of view of the symmetries, which reveals a manifestation of the influence of spatial topologies to dynamics.

4.1 Translation and Reflection Transformations

All symmetries concerning the Gowdy equations (92) should have their origin in the metric (85). For example, this metric is invariant under the diffeomorphism $x \to x - a$, *up to* the redefinition of the metric functions $(\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(x+a), \bar{Q}(x+a))$ (with the similar one for $\bar{\lambda}$). (For simplicity, we omit writing the argument t in the metric functions.) This redefinition accounts for the translation symmetry mentioned above. Recall that, if a metric is a solution for the vacuum Einstein equation, then so is the metric induced from a (spatial) diffeomorphism. Thus, in general, if such a diffeomorphism leaves the metric (85) invariant up to a redefinition of the metric functions, this redefinition defines a symmetry for the Gowdy equations.

Consider a diffeomorphism $\phi:\ (x,y,z) \to (\phi_x(x,y,z), \phi_y(x,y,z), \phi_z(x,y,z))$ with t fixed. As remarked, ϕ must preserve the characteristic form of the metric (85) so that a redefinition of the metric functions can make the metric invariant. In particular, to preserve the isothermal form $e^{\bar{\lambda}/2}(-\mathrm{d}t^2 + \mathrm{d}x^2)$ of the metric we must have $\phi_x(x,y,z) = \pm x - a$. Moreover, $(y,z) \to (\phi_y, \phi_z)$ must be a linear transformation on each y-z plane. That is, we must have

$$\phi:\ (x, \mathbf{y}) \to (\pm x - a, L\mathbf{y}), \tag{42}$$

where $L \in \mathrm{SL}^{\langle 2\rangle}$ is a constant matrix and $\mathbf{y}$ is the column form vector of (y,z). Here, $\mathrm{SL}^{\langle 2\rangle} \equiv \{m \in \mathrm{GL}(2,\mathbf{R}) |\det(m) = \pm 1\}$. (We do not have to consider the translations in the y-z plane of the type $(x,y,z) \to (x, y + y_0, z + z_0)$, which are isometries for the metric (85), since these

give rise to no effect to the Gowdy equations.) We decompose the ϕ_x part into $x \to x - a$ and $x \to -x$, and will call symmetries involved with the former *(generalized) translation symmetries* and call symmetries involved with the latter *(generalized) reflection symmetries*. We can also consider symmetries generated by diffeomorphisms involved with the identity $x \to x$, but since this type of symmetry is of little interest from the dynamical point of view discussed in the next subsection, we do not consider them here. We can find all possible translation and reflection symmetries defined above by finding $\phi_y(y,z)$ and $\phi_z(y,z)$ (or L) for each case, as follows.

We first consider the translation symmetries. As we will see in the next subsection, this symmetry implies that (in the Hamiltonian picture) a symmetric initial data with respect to a (generalized) translation maintains its symmetry under the time evolution. At the (locally) homogeneous limit, this implies that the diffeomorphism that gives rise to this symmetry should coincide with an isometry for the metric. Recall that there is a three-dimensional isometry group G for each case of T^3, E^3, Nil, and Sol at the homogeneous limit. It contains a two-dimensional commutative isometry subgroup $H \simeq \mathbf{R}^2$ that is preserved as isometry group when the metric becomes $\mathcal{LU}^2$-symmetric but inhomogeneous. Then, there is a one-parameter isometry subgroup K which acts transversely to H. In the $\mathcal{LU}^2$-symmetric but inhomogeneous cases, the actions of K are served as the desired translations. As easily found from the multiplication rules for the Bianchi I, VII$_0$, II, and VI$_0$ groups, they are, respectively, given by

$$\begin{aligned}
T_a &: (x,y,z) \to (x-a,\, y,\, z), && (43)\\
E_a &: (x,y,z) \to (x-a,\, \cos \mathrm{q}a\, y + \sin \mathrm{q}a\, z,\, -\sin \mathrm{q}a\, y + \cos \mathrm{q}a\, z), && (44)\\
N_a &: (x,y,z) \to (x-a,\, y,\, z-ay), && (45)\\
S_a &: (x,y,z) \to (x-a,\, e^{2\mathrm{q}a} y,\, e^{-2\mathrm{q}a} z). && (46)
\end{aligned}$$

Since these leave the invariant one-forms for, respectively, Bianchi I, VII$_0$, II, and VI$_0$ types invariant, the corresponding transformations (redefinitions) takes the simplest form for the unbarred metric functions:

$$\begin{aligned}
T_{a*} &: (\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(x+a), \bar{Q}(x+a)) \quad \text{for metric functions in (85)}, && (47)\\
E_{a*} &: (P(x), Q(x)) \to (P(x+a), Q(x+a)) \quad \text{for metric functions in (32)}, && (48)\\
N_{a*} &: (P(x), Q(x)) \to (P(x+a), Q(x+a)) \quad \text{for metric functions in (18)}, && (49)\\
S_{a*} &: (P(x), Q(x)) \to (P(x+a), Q(x+a)) \quad \text{for metric functions in (27)}. && (50)
\end{aligned}$$

Here, we assume $a = 2\pi/n$ for a positive integer n. Using the relations (33), (19) and (29), E_{a*}, N_{a*} and S_{a*} expressed with the canonical metric (85) are found to be

$$\begin{aligned}
E_{a*} &: (\bar{P}(x), \bar{Q}(x)) \to (\, \log \Phi(\bar{P}(x+a), \bar{Q}(x+a), \cos \mathrm{q}a, -\sin \mathrm{q}a), \\
&\qquad\qquad \Psi(\bar{P}(x+a), \bar{Q}(x+a), \cos \mathrm{q}a, -\sin \mathrm{q}a)\,), && (51)\\
N_{a*} &: (\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(x+a), \bar{Q}(x+a) + a), && (52)\\
S_{a*} &: (\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(x+a) - 2\mathrm{q}a, e^{2\mathrm{q}a} \bar{Q}(x+a)). && (53)
\end{aligned}$$

The functions Φ and Ψ are defined in Eq.(34). T_{a*}, E_{a*}, N_{a*} and S_{a*} preserve the boundary conditions for the $\mathcal{LU}^2$-symmetric models of, respectively, T^3, E^3, Nil and Sol types, so they

define the translation symmetry for each type. Note that the translations for each type form the infinite cyclic group $\mathbf{Z}$ for a fixed a, or form the cyclic group $\mathbf{Z}_n$ for a fixed $a = 2\pi/n$ if the (periodic) boundary conditions are taken into account.

Next, we consider the reflection symmetries, by which we mean the symmetries generated by the reflection transformations defined as follows.

Definition Consider the map $R: \ (x, \mathbf{y}) \to (-x, L\mathbf{y})$ for a given $L \in \mathrm{SL}^{\langle 2 \rangle}$. The symmetry transformation R_* induced from R, acting on the variables (P, Q) of a given model, is called a *reflection transformation*, if the following three conditions are fulfilled;

(i) For R (and R_*) to form the $\mathbf{Z}_2$-group together with the identity id,

$$R^2 \equiv R \circ R = \mathrm{id}. \tag{54}$$

(ii) As explained later,

$$\mathrm{Conj}(R)T \equiv R \circ T \circ R^{-1} = T^{-1}, \tag{55}$$

where T is the map inducing the translation transformation of the given model. (Hence, T is one of Eqs.(43) $\sim$ (46).)

(iii) R_* preserves the boundary conditions. That is, if variables (P, Q) satisfy the boundary conditions appropriate for the model, $R_*(P, Q)$ also satisfy the same boundary conditions.

First, from condition (i), L must be one of

$$L_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_\pi \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ or } \quad L_{l_\theta} \equiv \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \tag{56}$$

where $\theta \in [0, \pi)$ is a real parameter. These act on the y-z plane as, respectively, the identity, the rotation by angle π, and the reflection with respect to the line l_θ which passes through the origin with angle θ. Thus we have obtained the following candidates for the reflection R:

$$R_0 \ : \ (x, y, z) \to (-x, y, z), \tag{57}$$

$$R_\pi \ : \ (x, y, z) \to (-x, -y, -z), \tag{58}$$

$$R_{l_\theta} \ : \ (x, y, z) \to (-x, y\cos 2\theta + z \sin 2\theta, y \sin 2\theta - z \cos 2\theta). \tag{59}$$

Condition (ii) concerns a "compatibility" with the natural translation obtained earlier. That is, the reflection about a generic point $x = a$ should be induced from that about $x = 0$ by a translation. We can equivalently require that the conjugation, $\mathrm{Conj}(R)T$, of the translation T by a reflection R reverse the original translation, as indicated by Eq.(55). For example, the conjugation of the translation T_a for the T^3 model by the map R_0 is given by $\mathrm{Conj}(R_0)T_a = R_0 \circ T_a \circ (R_0)^{-1} = R_0 \circ T_a \circ R_0 = T_{-a}$. This shows R_0 is compatible with T_a. On the other hand, say, the conjugation of N_a by R_0 gives $\mathrm{Conj}(R_0)N_a = R_0 \circ N_a \circ R_0 = T_{-2a} \circ N_a$, which is not the reverse of N_a. All the compatible relations are given as follows:

For T^3 model:

$$\mathrm{Conj}(R_0)T_a = T_{-a}, \tag{60}$$

$$\mathrm{Conj}(R_\pi)T_a = T_{-a}, \tag{61}$$

$$
\begin{aligned}
& & \mathrm{Conj}(R_{l_\theta})T_a = T_{-a} \quad (\theta \in [0,\pi)), & \qquad (62)\\
&\text{For } E^3 \text{ model:} & \mathrm{Conj}(R_{l_\theta})E_a = E_{-a} \quad (\theta \in [0,\pi)), & \qquad (63)\\
&\text{For Nil model:} & \mathrm{Conj}(R_{l_\theta})N_a = N_{-a} \quad (\theta = 0, \frac{\pi}{2}), & \qquad (64)\\
&\text{For Sol model:} & \mathrm{Conj}(R_{l_\theta})S_a = S_{-a} \quad (\theta = \frac{\pi}{4}, \frac{3}{4}\pi). & \qquad (65)
\end{aligned}
$$

The transformations for the metric functions induced from these maps are given as follows:

$$
\begin{aligned}
R_{0*},\, R_{\pi*} \ &: \ (\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(-x), \bar{Q}(-x)), \qquad (66)\\
R_{l_\theta *} \ &: \ (\bar{P}(x), \bar{Q}(x)) \to (\,\log\Phi(\bar{P}(-x), \bar{Q}(-x), -\sin 2\theta, \cos 2\theta),\\
&\qquad -\Psi(\bar{P}(-x), \bar{Q}(-x), -\sin 2\theta, \cos 2\theta)\,), \qquad (67)
\end{aligned}
$$

where the functions Φ and Ψ are defined in Eq.(34). The explicit forms for Nil ($\theta = 0, \pi/2$) and Sol ($\theta = \pi/4, 3\pi/4$) models are

$$
\begin{aligned}
R_{l_0 *},\, R_{l_{\pi/2} *} \ &: \ (\bar{P}(x), \bar{Q}(x)) \to (\bar{P}(-x), -\bar{Q}(-x)), \qquad (68)\\
R_{l_{\pi/4} *},\, R_{l_{3\pi/4} *} \ &: \ (\bar{P}(x), \bar{Q}(x)) \to \left(\bar{P} + \log(\bar{Q}^2 + e^{-2\bar{P}}),\ \frac{\bar{Q}}{\bar{Q}^2 + e^{-2\bar{P}}}\right)\Bigg|_{x\to -x}. \qquad (69)
\end{aligned}
$$

Note that $R_{0*} = R_{\pi*}$, $R_{l_0 *} = R_{l_{\pi/2} *}$, and $R_{l_{\pi/4} *} = R_{l_{3\pi/4} *}$.

As a final task we are left with checking condition (iii). For the T^3 model, we at once see that if $\bar{P}(x)$ and $\bar{Q}(x)$ are periodic functions, then their images by R_{0*} and $R_{l_\theta *}$ are also periodic functions. Thus, R_{0*} and $R_{l_\theta *}$ give reflections for the T^3 model. For the E^3, Nil and Sol models it is convenient to work with the unbarred variables. Interestingly the transformations (67) (for E^3), (68) (for Nil) and (69) (for Sol) are written with the unbarred variables in exactly the same form (except for the disappearance of bars);
For E^3:

$$
\begin{aligned}
R_{l_\theta *} \ &: \ (P(x), Q(x)) \to (\,\log\Phi(P(-x), Q(-x), -\sin 2\theta, \cos 2\theta),\\
&\qquad -\Psi(P(-x), Q(-x), -\sin 2\theta, \cos 2\theta)\,), \qquad (70)
\end{aligned}
$$

For Nil:

$$
R_{l_0 *} : \ (P(x), Q(x)) \to (P(-x), -Q(-x)), \qquad (71)
$$

For Sol:

$$
R_{l_{\pi/4} *} : \ (P(x), Q(x)) \to \left(P + \log(Q^2 + e^{-2P}),\ \frac{Q}{Q^2 + e^{-2P}}\right)\Bigg|_{x\to -x}. \qquad (72)
$$

From these it is trivial to see that the periodicity of the unbarred variables are preserved for E^3, Nil and Sol. This confirms that these transformations do give reflections for the corresponding models. Table 1 bellow summarizes the translation and reflection transformations for each model.

Remark that we have inclusion relations. The T^3 model has the largest set of reflections. The E^3 models have part of that for the T^3 model. The Nil and Sol models have parts of that for the E^3 models.

Model	Translation	Reflections
T^3	T_{a*}	$R_{0*},\ R_{l_\theta *}$
E^3	E_{a*}	$R_{l_\theta *}$
Nil	N_{a*}	$R_{l_0 *}$
Sol	S_{a*}	$R_{l_{\pi/4} *}$

Table 1: The translation and reflection symmetry transformations. For reflections of T^3 and E^3, $\theta \in [0, \pi)$.

4.2 Dynamical Interpretation

As well known [2], the Gowdy equations (92) admit a Hamiltonian formulation. Let $\pi_P(x)$ and $\pi_Q(x)$ be, respectively, conjugate momenta of $P(x)$ and $Q(x)$. The phase space $\mathcal{P}$ is spanned by the four functions $(P(x), Q(x), \pi_P(x), \pi_Q(x))$ (or similarly for variables with bars). Consider the set $\mathcal{O}$ of maps on $\mathcal{P}$, $\mathcal{O} \equiv \{o | o : \mathcal{P} \to \mathcal{P}\}$. The Hamiltonian flow ψ_τ, which generates the time evolution by time τ, is naturally regarded as an element in O, $\psi_\tau \in \mathcal{O}$, for a fixed τ. ψ_0 is supposed to be the identity.

What we are interested in here are maps (operators) which commute with ψ_τ, so we define

$$\mathcal{O}_{\rm com} \equiv \{ f \in \mathcal{O} |\ {}^\forall \tau \in \mathbf{R}, \quad f \circ \psi_\tau = \psi_\tau \circ f \}. \tag{73}$$

We call an $f \in \mathcal{O}_{\rm com}$ a *symmetry operator*. (If the operator f was a smooth flow like ψ_τ we would be able to reformulate this commutativity to the vanishing of a Poisson bracket. However, since we consider discrete operators as f this is not the case. As a result, we do not actually need symplectic structures if only the flow ψ_τ is properly defined.) For an initial data $\gamma \in \mathcal{P}$, $\psi_\tau(\gamma)$ is a solution for the Einstein equation as a function of τ. If there is a symmetry operator f, then the function of τ, $f(\psi_\tau(\gamma))$, is also a solution, since this equals to $\psi_\tau(f(\gamma))$, which is the solution with the initial data $f(\gamma)$. Therefore a symmetry operator generates a (in general, distinct) solution from a solution.

Now, note that, while a symmetry transformation obtained in the previous subsection is defined in the configuration space spanned by P and Q, it induces an operator on the phase space $\mathcal{P}$ by differentiating the configuration variables with respect to τ. We use in particular the Hamiltonian equations

$$\dot{P} = \pi_P, \quad \dot{Q} = e^{-2P} \pi_Q. \tag{74}$$

(This relation is the same for all sets of barred or unbarred variables.) Thus, the symmetry transformations are also regarded as an operator on $\mathcal{P}$. This operator is clearly a symmetry operator. Conversely, if there is a symmetry operator, it gives a symmetry transformation, as seen from the previous paragraph. Thus, in effect, symmetry transformations and operators are equal entities. Note that, since we have exhausted all the symmetry transformations concerning translation and reflection, we now know all the symmetry operators for them.

To represent the symmetry operator obtained from a symmetry transformation we use the same symbol as that of the symmetry transformation. For example, the translation transformation T_{a*} for T^3 model defined in Eq.(47) naturally gives the operator, denoted also as T_{a*}, that

simply shifts the argument by a constant:

$$T_{a*}:(\bar{P}(x),\bar{Q}(x),\bar{\pi}_P(x),\bar{\pi}_Q(x))\to(\bar{P}(x+a),\bar{Q}(x+a),\bar{\pi}_P(x+a),\bar{\pi}_Q(x+a)). \tag{75}$$

The translation symmetry operators E_{a*} (for E^3), N_{a*} (for Nil) and S_{a*} (for Sol) are also defined by similar maps if the phase space variables are taken as the "unbarred" ones. (cf. Eqs.(48), (49), and (50).)

An interesting consequence of the symmetry operators comes from considering data which are invariant with respect to this operator. For $f\in\mathcal{O}_{\rm com}$, let

$$\mathcal{S}(f)\equiv\{\gamma\in\mathcal{P}|f(\gamma)=\gamma\}. \tag{76}$$

For example, if f is a translation symmetry operator $T_{2\pi/n*}$, the set $\mathcal{S}(T_{2\pi/n*})$ is comprised of data in $\mathcal{P}$ which are translation-symmetric with period $2\pi/n$. If f is a reflection symmetry operator R_*, the set $\mathcal{S}(R_*)$ is comprised of reflection-symmetric data in $\mathcal{P}$. The remarkable fact is that *these symmetries are preserved dynamically*, since Eq.(73), together with $f(\gamma)=\gamma$, implies $f(\psi_\tau(\gamma))=\psi_\tau(\gamma)$ for all τ. In other words, any data $\gamma\in\mathcal{S}(f)$ remains within $\mathcal{S}(f)$ under the dynamical flow, i.e., $\psi_\tau(\mathcal{S}(f))\subseteq\mathcal{S}(f)$. Thus, $\mathcal{S}(f)$ defines an *invariant subset* of $\mathcal{P}$ for the flow ψ_τ.

We interpret the invariant subset $\mathcal{S}(f)$ as a dynamical character of the model. Motivated by this interpretation, we compare $\mathcal{S}(f)$ to see if there are distinctions in dynamical characters of the models. To this, recall that if we think of the phase space $\mathcal{P}$ as being spanned by the unbarred variables, $\mathcal{P}$ for every model is a space of four functions which are all periodic functions. (We think that the barred and unbarred variables coincide for the T^3 model.) Therefore the phase spaces for the four models are naturally identified. In this view, the flow ψ_τ depends on the type of the model. Strictly speaking, however, since the "constraints" equations corresponding to Eq.(95) take different forms for the E^3, Nil and Sol models, this identification may be justified only approximately, but we neglect this point.

First, consider the translations. As already mentioned the translation operator for every model is obtained by simply shifting the spatial coordinate like Eq.(75) for the unbarred variables. Hence, all the invariant subsets $\mathcal{S}(T_{a*})$, $\mathcal{S}(E_{a*})$, $\mathcal{S}(N_{a*})$, and $\mathcal{S}(S_{a*})$ for $a=2\pi/n$, where n is a positive integer, are spanned by periodic data with period $2\pi/n$. In this sense, we roughly represent

$$\mathcal{S}(T_{a*})\simeq\mathcal{S}(E_{a*})\simeq\mathcal{S}(N_{a*})\simeq\mathcal{S}(S_{a*}). \tag{77}$$

Therefore we conclude that for the property of preservation of translation symmetry, there is no significant distinction between the four models.

Our main interest is therefore in the reflections. As we remarked, reflection operator

$$R_{0*}:\ (\bar{P}(x),\bar{Q}(x),\bar{\pi}_P(x),\bar{\pi}_Q(x))\to(\bar{P}(-x),\bar{Q}(-x),\bar{\pi}_P(-x),\bar{\pi}_Q(-x)) \tag{78}$$

does not preserve the boundary conditions for E^3, Nil and Sol models, so that it is not relevant for these three models. At first sight, the similar operator acting on unbarred variables

$$R'_{0*}:\ (P(x),Q(x),\pi_P(x),\pi_Q(x))\to(P(-x),Q(-x),\pi_P(-x),\pi_Q(-x)), \tag{79}$$

which does preserve the boundary conditions for the three models, seems to be a natural reflection operator for them. Indeed, we can regard the spatial coordinate x for the unbarred variables as a natural coordinate in view of the translation property shown above. That is, the reflection with respect to this coordinate seems like the most natural one. Nevertheless, the reflection symmetry with respect to this operator imposed on an initial data for E^3, Nil or Sol model is *not preserved* under the time evolution. This fact can be seen directly from the dynamical equations for the unbarred variables. For example, those for the Nil model are given by (See Eq.(19))

$$\begin{aligned} \ddot{P} &- e^{-2\tau}P'' - e^{2P}(\dot{Q}^2 - e^{-2\tau}(Q'-1)^2) = 0, \\ \ddot{Q} &- e^{-2\tau}Q'' + 2(\dot{P}\dot{Q} - e^{-2\tau}P'(Q'-1)) = 0, \end{aligned} \tag{80}$$

which are apparently, in contrast to the T^3 model, not invariant under the transformation $\partial/\partial x \to -\partial/\partial x$, due to the factor $(Q'-1)$. Similarly, those for the E^3 and Sol models are not invariant under the same transformation. As a result, in contrast to translation, there are no invariant subsets for reflection in the E^3, Nil and Sol models which naturally correspond to the invariant subset $\mathcal{S}(R_{0*})$ for the T^3 model.

However, the T^3 model admits another one-parameter family of nontrivial reflection operators $R_{l_\theta *}$. (See Eq.(67)) The momentum part is derived from Eq.(67) by differentiating with respect to τ and using Eqs.(74). The result is given by

$$\begin{aligned} R_{l_\theta *}:\ (\bar{\pi}_P(x), \bar{\pi}_Q(x)) \ \to\ & \Bigg(\frac{((c+\bar{Q}s)^2 - e^{-2\bar{P}}s^2)\bar{\pi}_P + 2e^{-2\bar{P}}s(c+\bar{Q}s)\bar{\pi}_Q}{(c+\bar{Q}s)^2 + e^{-2\bar{P}}s^2}, \\ & 2s(c+\bar{Q}s)\bar{\pi}_P - ((c+\bar{Q}s)^2 - e^{-2\bar{P}}s^2)\bar{\pi}_Q \Bigg)\Bigg|_{x\to -x}, \end{aligned} \tag{81}$$

where $c \equiv \cos 2\theta,\ s \equiv \sin 2\theta$.

As shown in the previous subsection, for Nil and Sol models only isolated values ($\theta = 0$ or $\pi/2$ for Nil, and $\pi/4$ or $3\pi/4$ for Sol) of θ are permissible. As a result, the invariant subsets for reflection in the Nil and Sol models are much limited compared to the T^3 model. More precisely, we have the following.

Proposition 9 *Let $\mathcal{S}(R_T)$, $\mathcal{S}(R_E)$, $\mathcal{S}(R_N)$, and $\mathcal{S}(R_S)$ be the unions of the all invariant subsets for reflection in, respectively, the T^3, E^3, Nil, and Sol models. Then, the inclusion relation*

$$\mathcal{S}(R_T) \supset \mathcal{S}(R_E) \supset (\mathcal{S}(R_N),\ \mathcal{S}(R_S)) \tag{82}$$

holds (if neglecting the constraints corresponding to Eq.(95)).

Proof: This can be seen from $\mathcal{S}(R_T) = \mathcal{S}(R_{0*}) \cup (\cup_\theta \mathcal{S}(R_{l_{\theta *}}))$, $\mathcal{S}(R_E) = \cup_\theta \mathcal{S}(R_{l_{\theta *}})$, $\mathcal{S}(R_N) = \mathcal{S}(R_{l_{0*}})$, and $\mathcal{S}(R_E) = \mathcal{S}(R_{l_{\pi/4 *}})$. □

It should be stressed that the relation (82) truly reflects the dynamical properties of the models. Note that, since the (periodic) boundary conditions imposed on initial data are the same for all models if the phase space $\mathcal{P}$ is spanned by the unbarred variables, a map f which preserves the boundary conditions for a model always preserves the boundary conditions for

other models, too. This means that there are the same amounts of "reflection symmetric initial data" for the four models. However, the time evolution does not always preserve the symmetry of such an initial data, as we have seen. To conclude, we have clarified how much "reflection" symmetric data exist *for which the symmetry is preserved* under the time evolution for each model, and found, in particular, the inclusion relation (82). This may be interpreted as a manifestation of the influence of topology to dynamics.

5 Summary and Comments

We have made the structures of $\mathcal{LU}^2$-manifolds clear and classified the possible topologies of them. For convenience we have split the set of these manifolds into two kinds; *the first kind*, those with local Killing vectors which are not degenerate everywhere, and *the second kind*, those with ones which are degenerate somewhere. The local Killing vectors of any $\mathcal{LU}^2$-manifold of the second kind are actually defined globally, so that all the $\mathcal{LU}^2$-symmetric models of the second kind are contained in the usual Gowdy models. On the other hand, it is only T^3 that is contained in the Gowdy models among the varieties of the $\mathcal{LU}^2$-manifolds of the first kind, so that in this paper we have basically restricted ourselves to the models of the first kind.

In the case that three-manifold M is an $\mathcal{LU}^2$-symmetric space of the first kind, M is a T^2-bundle over the S^1 (Theorem 2), and according to the corresponding geometric structure, M is naturally characterized by one of E^3, Nil, or Sol. In each case, the possible topologies of M can be classified more precisely, but if we are interested in the dynamical properties of the corresponding spacetime models it is not necessary to consider all the models one by one. Restricted to representatives for this "dynamical equivalence class," the number of models to consider is to great extent decreased. In particular, there is three representatives for E^3 (cf. Sec.3.3), and there is only one for Nil (cf. Sec.3.1). The representatives for Sol are parameterized by only one discrete parameter q (cf. Sec.3.2).

We have given two ways of representing the metric for each of E^3, Nil and Sol models. (We distinguish between the E^3 and T^3 models. See **Convention** in Sec.3.3.) Note that the metrics of all the $\mathcal{LU}^2$-symmetric spacetimes can be represented locally in the same canonical form, but with distinct boundary conditions. In this view, however, global symmetries that arise from the spatial topology tend to be unclear. On the other hand, the other way of representing the metric makes the geometric structure the $\mathcal{LU}^2$-manifold M admits manifest. Geometrically it is obtained by "relaxing" the corresponding locally homogeneous metric. This type of metric has another advantage that it gives rise to a natural reduction of the model; that is, since the boundary conditions are always given by the periodicity for the spatial coordinate, the spatial manifold naturally reduces to the S^1, the base space of the bundle. Note that this fact makes the identifications of the phase spaces for the varieties of the models possible.

Finally, as an application of these fundamental facts we have given the translation and reflection operators which commute with the time-evolution, and have discussed their significance. Since, as stated above, the metrics of $\mathcal{LU}^2$-symmetric models can always be locally represented in the same form, these models are considered to have the same *local* dynamical properties. However differences of topologies affect the *global* dynamical properties of the models through the differences of the boundary conditions. In this paper we have examined whether or not a

global symmetry imposed on an initial data is preserved in time, or how much there exist global symmetries which are preserved in time, for each model of the first kind. As a result, we have indeed found that there are remarkable distinctions in the properties of *reflection*. The freedom of reflection symmetries is the largest as for the T^3 model, and we have the inclusion relation (82). In particular, naive (even type) reflection symmetric initial data for the E^3, Nil or Sol model do not evolve maintaining its symmetry in time, in contrast to the T^3 model.

In the following we make some comments. A first one concerns another point concerning the correlations between topology and dynamics. As for the dynamics of the Gowdy T^3 model, what has been being taken an interest most is whether the AVTD conjecture [18] is correct or not that predicts a universal behavior of the approach toward the initial singularity. This conjecture has been basically supported both analytically [19, 20] and numerically [21, 22]. However, the subtlety has been pointed out [21] that there is a measure-zero set of spatial points where the AVTD behavior is not achieved. Here we concern ourselves with these *nongeneric* points. It is known [21] that at these points $\bar{Q}' = 0$. Since this condition is local, it does not depend on the topology. However, note that while points of $\bar{Q}' = 0$ are inevitable in the T^3 model, since $\bar{Q}(x)$ is a (smooth) periodic function, such points are not necessary in the Nil and Sol models. Hence we can naturally expect that topology affects the tendency of the appearance of the nongeneric points. The points where $\bar{Q}' = 0$ correspond, from Eqs.(19) and (29), to the points where

$$\text{Nil:} \qquad Q' = 1, \tag{83}$$

$$\text{Sol:} \qquad (\log|Q|)' = 2\mathrm{q}, \tag{84}$$

for the unbarred variables. The unbarred $Q(x)$ must be periodic, but this condition does not force the existence of points such that the above conditions are fulfilled. However, since the evolution of Q freezes when approaching the initial singularity, we can naturally expect that nongeneric points are still generated if the maximal gradient of Q at an initial surface is large enough and if, as a result, $Q'_{\max} > 1$ (Nil), or $(\log|Q|)'_{\max} > 2\mathrm{q}$ (Sol) at a time when the evolution of Q freezes. Indeed, this property has been observed from numerical simulations the author performed. A detailed description for this point will be reported elsewhere. In Ref. [3] it was suggested that the nongeneric points are not generated in the Nil and Sol models, since the conditions corresponding to Eqs.(84) and (84) were written in such different forms with different choice of metric functions that the conditions seem not to be fulfilled. This claim is not correct, as seen from above. Still, we may claim that the E^3 model is most likely to generate nongeneric points.

Next we comment on the locally U(1)-symmetric models [23] that are less symmetric than the $\mathcal{LU}^2$-symmetric models. This model admits only one spatial local Killing vector. The spatial manifold M is again assumed to be closed. First, from an analogous analysis to the proof of Lemma 1, the Killing orbits are found to be closed and are homeomorphic to S^1, if the local Killing vector does not degenerate everywhere (that is, if the model is the "first kind"). Hence M is naturally a Seifert fiber space [17], and still admits a geometric structure, which is one of E^3, Nil, $H^2 \times \mathbf{R}$, $\mathrm{SL}(2, \mathbf{R})$, $S^2 \times \mathbf{R}$, or S^3 [13]. In the case that M is the "second kind", i.e., the local Killing vector vanishes on somewhere, the manifold M is obtained from a Seifert fiber space by performing a finite number of particular kind of Dehn surgeries. The resulting manifold M is found (from one in the series of famous theorems of Thurston concerning the

H^3 structure) to again admit a geometric structure. Detailed accounts and applications will be presented elsewhere.

A Vacuum Einstein equations for Gowdy Spacetimes

In this Appendix, we summarize the standard prescription [2] of the reduced vacuum Einstein equations for Gowdy spacetimes, together with generalizations to Nil and Sol types.

We consider the two-surface orthogonal metric (12). This is used most frequently in the literature since the Einstein equations become to great extent simpler.

More explicitly, we write [24]

$$\mathrm{d}s^2 = e^{\bar{\gamma}/2}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \bar{R}(e^{\bar{P}}(\mathrm{d}y + \bar{Q}\mathrm{d}z)^2 + e^{-\bar{P}}\mathrm{d}z^2). \tag{85}$$

where $\bar{P}$ and $\bar{Q}$ are functions of t and x. Then, the vacuum Einstein equations for the metric functions are explicitly given as follows. The dynamical equations are

$$\begin{aligned}\bar{P}_{tt} &- \bar{P}'' - e^{2\bar{P}}(\bar{Q}_t^2 - \bar{Q}'^2) + \bar{R}^{-1}(\bar{R}_t\bar{P}_t - \bar{R}'\bar{P}') = 0,\\ \bar{Q}_{tt} &- \bar{Q}'' + 2(\bar{P}_t\bar{Q}_t - \bar{P}'\bar{Q}') + \bar{R}^{-1}(\bar{R}_t\bar{Q}_t - \bar{R}'\bar{Q}') = 0,\end{aligned} \tag{86}$$

and

$$\bar{R}_{tt} - \bar{R}'' = 0, \tag{87}$$

where subscript t represents the derivative with respect to t and the primes represent the derivatives with respect to x. The remaining constraint equations takes a simple form in the null coordinates $u = t - x$ and $v = t + x$;

$$\begin{aligned}\bar{R}(\bar{P}_u^2 + e^{2\bar{P}}\bar{Q}_u^2) &= -2\bar{R}_{uu} + \bar{R}_u(\bar{\gamma}_u + \bar{R}^{-1}\bar{R}_u),\\ \bar{R}(\bar{P}_v^2 + e^{2\bar{P}}\bar{Q}_v^2) &= -2\bar{R}_{vv} + \bar{R}_v(\bar{\gamma}_v + \bar{R}^{-1}\bar{R}_v),\end{aligned} \tag{88}$$

where the subscripts stand for the derivatives thereof.

The last two equations can be solved for $\bar{\gamma}_u$ and $\bar{\gamma}_v$ at any spacetime point wherever $\bar{R}_u$ and $\bar{R}_v$ do not vanish. In this case the integrability condition for $\bar{\gamma}$, namely $\bar{\gamma}_{uv} - \bar{\gamma}_{vu} = 0$, is automatically satisfied if the dynamical equations (86) are satisfied. Hence we can integrate the equations and obtain $\bar{\gamma}$ by using solutions of the (unconstrained) dynamical equations (86). If there is a spacetime point such that $\bar{R}_u = 0$ (or $\bar{R}_v = 0$), Eqs.(88) cannot be solved for $\bar{\gamma}_u$ (or $\bar{\gamma}_v$), so that the constraint equation (88) constrains $\bar{P}$ and $\bar{Q}$ at that spacetime point. This condition is called the "matching condition" [2]. Explicitly,

$$\begin{aligned}\bar{P}_u^2 + e^{2\bar{P}}\bar{Q}_u^2 &= -2\bar{R}^{-1}\bar{R}_{uu} \quad \text{at} \quad \bar{R}_u = 0,\\ \bar{P}_v^2 + e^{2\bar{P}}\bar{Q}_v^2 &= -2\bar{R}^{-1}\bar{R}_{vv} \quad \text{at} \quad \bar{R}_v = 0.\end{aligned} \tag{89}$$

From the observation that the left hand sides of Eqs.(89) are positive semidefinite, we find that for $R > 0$,

$$\begin{aligned}\bar{R}_{uu} \leq 0 \quad &\text{at} \quad \bar{R}_u = 0,\\ \bar{R}_{vv} \leq 0 \quad &\text{at} \quad \bar{R}_v = 0.\end{aligned} \tag{90}$$

An implication of these inequalities is called the "corner theorem" [2]. Here, a *corner* is a point in the t-x plane such that $\bar{R}_u = 0$ or $\bar{R}_v = 0$, that is, a point where the gradient of a level curve of $\bar{R}$ becomes null. The implication of the inequalities would be clear if observing some level curves of $\bar{R}$ near a corner in the u-v coordinates. It should be stressed that these inequalities do *not* depend on the boundary conditions for $\bar{P}$, $\bar{Q}$, and $\bar{\gamma}$, so hold for any spatial topology. (This theorem, however, tacitly assume the finiteness of $\bar{\gamma}_u$ and $\bar{\gamma}_v$ at the corner. If we allow for $\bar{R}_u\bar{\gamma}_u$ and $\bar{R}_v\bar{\gamma}_v$ to remain finite at the corner with diverging $\bar{\gamma}_u$ and $\bar{\gamma}_v$, we may have counterexamples to this theorem. [25])

For any $\mathcal{LU}^2$-symmetric spacetime of the first kind, the area function $\bar{R}$ must be periodic (cf. Eq.(13)). Hence, corners appear only in pair, implying that $\bar{R}$ cannot have a corner. We can therefore choose $\bar{R}$ spatially constant

$$\bar{R}(t,x) = t, \tag{91}$$

(using the remaining freedom of the coordinate transformations of type $u \to F(u), v \to G(v)$). In this case, the reduced Einstein equations (86) and (88) become

$$\begin{aligned} \ddot{\bar{P}} \ &- \ e^{-2\tau}\bar{P}'' - e^{2\bar{P}}(\dot{\bar{Q}}^2 - e^{-2\tau}\bar{Q}'^2) = 0, \\ \ddot{\bar{Q}} \ &- \ e^{-2\tau}\bar{Q}'' + 2(\dot{\bar{P}}\dot{\bar{Q}} - e^{-2\tau}\bar{P}'\bar{Q}') = 0, \end{aligned} \tag{92}$$

with

$$\begin{aligned} &\dot{\bar{\lambda}} = \dot{\bar{P}}^2 + e^{-2\tau}\bar{P}'' + e^{2\bar{P}}(\dot{\bar{Q}}^2 + e^{-2\tau}\bar{Q}'^2), \\ &\bar{\lambda}' = 2(\bar{P}'\dot{\bar{P}} + e^{2\bar{P}}\bar{Q}'\dot{\bar{Q}}). \end{aligned} \tag{93}$$

Here, we have put

$$t = e^{-\tau}, \quad e^{\bar{\gamma}/2} = e^{-\bar{\lambda}/2+\tau/2}. \tag{94}$$

Dots in these equations represent derivatives with respect to τ (not t). $\bar{\lambda}$ should be periodic for any $\mathcal{LU}^2$-symmetric spacetime of the first kind. This leads to the constraint for $\bar{P}$ and $\bar{Q}$,

$$\int_0^{2\pi} (\bar{P}'\dot{\bar{P}} + e^{2\bar{P}}\bar{Q}'\dot{\bar{Q}})\mathrm{d}x = 0, \tag{95}$$

because of the second equation of Eqs.(93).

The functions $\bar{P}$ and $\bar{Q}$ are not periodic functions for the Nil and Sol models. Appropriate boundary conditions are obtained from Eqs.(19) and (29). Specifically,
For Nil:

$$\bar{P}(x+2\pi) = \bar{P}(x), \ \bar{Q}(x+2\pi) = \bar{Q}(x) - 2\pi. \tag{96}$$

For Sol:

$$\bar{P}(x+2\pi) = \bar{P}(x) + 4\pi \mathrm{q}, \ \bar{Q}(x+2\pi) = e^{-4\pi \mathrm{q}}\bar{Q}(x). \tag{97}$$

Remark The last boundary conditions (96) and (97) do not necessarily coincide with those obtained from Eqs.(13). This is because the choice of the covering group Γ (cf. Eq.(10)) is different. In obtaining Eqs.(13) we have fixed Γ first and found the appropriate metric, but for Eqs.(96) and (97) we try to fix the universal covering metric with appropriate Γ as long as possible. This is the essential point in the dynamical equivalences for the Nil and Sol models presented in Sec.3.

B Compactification of Sol

In this Appendix we show calculations needed to perform the compactification of Sol. This is basically a review of that presented in Ref. [9]. In this reference, the form of $\mathrm{mcg}_+(T^2) \simeq \mathrm{SL}(2,\mathbf{Z})$ was restricted to that of Eq.(9), and the parameter q (See bellow) was not introduced. In this review we show an explicit calculation with general form of the matrix and with q, though no essential difference appears.

What we do is to embed the fundamental groups into Sol$\simeq G_{\mathrm{VI}_0}$ (the Bianchi VI_0 group). For notational convenience, letting $a = g_2$, $b = g_3$, and $c = g_1$ in the representation (6),

$$\pi_1 = \left\langle a, b, c;\ [a,b] = 1, cac^{-1} = a^p b^r, cbc^{-1} = a^q b^s \right\rangle, \tag{98}$$

where

$$A \equiv \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z}), \quad \text{with} \quad |\mathrm{Tr}A| > 2. \tag{99}$$

We represent the π_1-generators with their components in G_{VI_0}

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},\ b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \tag{100}$$

To avoid confusions with powers, we use subscripts to distinguish each components as above. The multiplication rule for G_{VI_0} is given by

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha + x \\ \beta + e^{-\mathrm{q}\alpha} y \\ \gamma + e^{\mathrm{q}\alpha} z \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^{-1} = \begin{pmatrix} -\alpha \\ -e^{\mathrm{q}\alpha}\beta \\ -e^{-\mathrm{q}\alpha}\gamma \end{pmatrix}, \tag{101}$$

where $\mathrm{q} > 0$ is a parameter (, which has nothing to do with the elements in the matrix A). Left invariant one-forms are then given by Eq.(26).

First, note the first components of the relations

$$cac^{-1} = a^p b^r,\ cbc^{-1} = a^q b^s. \tag{102}$$

Noting that the first component of the product of two elements of Sol is simply given by the sum of the first components of the elements, we obtain $a_1 = pa_1 + rb_1$ and $b_1 = qa_1 + sb_1$, that is,

$$\begin{pmatrix} p-1 & r \\ q & s-1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{103}$$

The determinant for the matrix appearing in the left hand side is given by $\det A - \mathrm{Tr}A = 1 - \mathrm{Tr}A$, which is less than -1 (when $\mathrm{Tr}A > 2$) or greater than 3 (when $\mathrm{Tr}A < -2$), from the assumptions on the matrix A. Hence, the matrix has its inverse, implying

$$a_1 = b_1 = 0. \tag{104}$$

Under this condition the relation $[a, b] = 1$ becomes trivial. Moreover, we can at once calculate

$$a^p = \begin{pmatrix} 0 \\ pa_2 \\ pa_3 \end{pmatrix}, \; b^r = \begin{pmatrix} 0 \\ rb_2 \\ rb_3 \end{pmatrix}, \; cac^{-1} = \begin{pmatrix} 0 \\ a_2 e^{-\mathrm{q}c_1} \\ a_3 e^{\mathrm{q}c_1} \end{pmatrix}, \text{ etc.,} \tag{105}$$

so that the second and third components of the relations (102) are given by

$$A \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = e^{\mathrm{q}c_1} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \quad A \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = e^{-\mathrm{q}c_1} \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}. \tag{106}$$

These reveal that $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $\begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$ are eigenvectors of A, and $e^{\pm \mathrm{q}c_1}$ are the eigenvalues. The characteristic polynomial is $\lambda^2 - \mathrm{Tr}A\,\lambda + \det A = 0$, or equivalently,

$$\lambda^2 - \mathrm{Tr}A\,\lambda + 1 = 0. \tag{107}$$

Since the eigenvalues $e^{\pm \mathrm{q}c_1}$ must be positive, the embedding is found to be possible only for the case $\mathrm{Tr}A > 2$. However, even for the case $\mathrm{Tr}A < -2$, if the map

$$h: \; (x, y, z) \to (x, -y, -z) \tag{108}$$

can be regarded as an isometry, we can perform the embedding by putting $c = h \circ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, where the column vector in the right hand side is an element of Sol, and $\circ$ represents the composition of maps. It is an easy task to retrace the calculation above. We will find in particular that the eigenvalues of A must equal to $-e^{\pm \mathrm{q}c_1}$ rather than $e^{\pm \mathrm{q}c_1}$.

Finally, the formal solutions for embedding are as follows.

(i) Case of $\mathrm{Tr}A > 2$: Letting the eigensystems (the pairs of an eigenvalue and the corresponding normalized eigenvector) for the matrix A be

$$\left\{ e^{\mathrm{q}c_1}, \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \right\}, \quad \left\{ e^{-\mathrm{q}c_1}, \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \right\}, \tag{109}$$

the solution is

$$a = \begin{pmatrix} 0 \\ \beta u_2 \\ \gamma v_2 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ \beta u_3 \\ \gamma v_3 \end{pmatrix}, \; c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{110}$$

where β, γ, c_2, and c_3 are parameters. Note that, while the product qc_1 is directly connected with the eigenvalues of A, we can freely specify one of q or c_1.

(ii) Case of $\mathrm{Tr}A < -2$: Letting the eigensystems be

$$\left\{ -e^{\mathrm{q}c_1}, \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \right\}, \quad \left\{ -e^{-\mathrm{q}c_1}, \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \right\}, \tag{111}$$

the solution is

$$a = \begin{pmatrix} \beta u_2 \\ \gamma v_2 \\ 0 \end{pmatrix}, \; b = \begin{pmatrix} \beta u_3 \\ \gamma v_3 \\ 0 \end{pmatrix}, \; c = h \circ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{112}$$

where β, γ, c_2 and c_3 are parameters. As in the case (i), one of q or c_1 can be freely specified.

Remark We have seen that the fundamental group for $\mathrm{Tr}A > 2$ can be embedded into G_{VI_0}. Also, we have seen that the fundamental group for $\mathrm{Tr}A < -2$ can be embedded into $G^{\langle 2\rangle}_{\mathrm{VI}_0}$, the group generated by G_{VI_0} and the $\mathbf{Z}_2$-factor h. It is easy to see that the same groups can be embedded into the smaller group $H_{\mathrm{VI}_0} \subset G_{\mathrm{VI}_0}$ which is given by making the first component of G_{VI_0} discrete like $c_1\mathbf{Z}$, or into $H^{\langle 2\rangle}_{\mathrm{VI}_0}$, where $H^{\langle 2\rangle}_{\mathrm{VI}_0} \subset G^{\langle 2\rangle}_{\mathrm{VI}_0}$ is the group generated by H_{VI_0} and the map h. Since $H^{\langle 2\rangle}_{\mathrm{VI}_0}$ is the isometry group of the universal covering spacetime (27) (with the appropriate choice of q, as shown in Sec.3.2), the same embeddings are possible for this (inhomogeneous) spacetime.

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